Kosovo National Mathematical Olympiad 2011

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– Grade 9

1 Let $x=\left(1+\frac{1}{n}\right)^n$ and $y=\left(1+\frac{1}{n}\right)^{n+1}$ where $n\in\mathbb{N}$. Which one of the numbers x^y, y^x is bigger ?

2 It is given the function $f:(\mathbb{R}-\{0\})\to\mathbb{R}$ such that $f(x)=x+\frac{1}{x}$.
Is this function injective ? Justify your answer.

3 A little boy wrote the numbers $1,2,\cdots,2011$ on a blackboard. He picks any two numbers x,y, erases them with a sponge and writes the number $|x-y|$. This process continues until only one number is left. Prove that the number left is even.

4 In triangle ABC medians of triangle BE and AD are perpendicular to each other. Find the length of $\overline{AB}$, if $\overline{BC}=6$ and $\overline{AC}=8$

5 Let $n>1$ be an integer and S_n the set of all permutations $\pi:\{1,2,\cdots,n\}\to\{1,2,\cdots,n\}$ where π is bijective function. For every permutation $\pi\in S_n$ we define:

$$F(\pi)=\sum_{k=1}^n|k-\pi(k)| \text{ and } M_n=\frac{1}{n!}\sum_{\pi\in S_n}F(\pi)$$

where M_n is taken with all permutations $\pi\in S_n$. Calculate the sum M_n.

– Grade 10

1 Suppose that the roots p,q of the equation $x^2-x+c=0$ where $c\in\mathbb{R}$, are rational numbers. Prove that the roots of the equation $x^2+px-q=0$ are also rational numbers.

2 Find all solutions to the equation:

$$\left(\left\lfloor x+\frac{7}{3}\right\rfloor\right)^2-\left\lfloor x-\frac{9}{4}\right\rfloor=16$$

3 Prove that the following inequality holds:

$$(\log_{24}48)^2+(\log_{12}54)^2>4$$

4 Let a, b, c be the sides of a triangle, and S its area. Prove:

$$a^2+b^2+c^2 \geq 4S\sqrt{3}$$

In what case does equality hold?

5 Let $n>1$ be an integer and S_n the set of all permutations $\pi : \{1,2,\cdots,n\} \to \{1,2,\cdots,n\}$ where π is bijective function. For every permutation $\pi \in S_n$ we define:

$$F(\pi)=\sum_{k=1}^n |k-\pi(k)| \text{ and } M_n=\frac{1}{n!}\sum_{\pi \in S_n} F(\pi)$$

where M_n is taken with all permutations $\pi \in S_n$. Calculate the sum M_n.

– Grade 11

1 It is given the function $f:\mathbb{R}\to\mathbb{R}$ such that it holds $f(\sin x)=\sin(2011x)$. Find the value of $f(\cos x)$.

2 Is it possible that by using the following transformations:

$$f(x)\mapsto x^2\cdot f\left(\frac{1}{x}+1\right) \text{ or } f(x)\mapsto (x-1)^2\cdot f\left(\frac{1}{x-1}\right)$$

the function $f(x)=x^2+5x+4$ is sent to the function $g(x)=x^2+10x+8$?

3 Find maximal value of the function $f(x)=8-3\sin^2(3x)+6\sin(6x)$

4 A point P is given in the square $ABCD$ such that $\overline{PA}=3$, $\overline{PB}=7$ and $\overline{PD}=5$. Find the area of the square.

5 Let $n>1$ be an integer and S_n the set of all permutations $\pi : \{1,2,\cdots,n\} \to \{1,2,\cdots,n\}$ where π is bijective function. For every permutation $\pi \in S_n$ we define:

$$F(\pi)=\sum_{k=1}^n |k-\pi(k)| \text{ and } M_n=\frac{1}{n!}\sum_{\pi \in S_n} F(\pi)$$

where M_n is taken with all permutations $\pi \in S_n$. Calculate the sum M_n.

– Grade 12

1 The complex numbers z_1 and z_2 are given such that $z_1 = -1 + i$ and $z_2 = 2 + 4i$. Find the complex number z_3 such that z_1, z_2, z_3 are the points of an equilateral triangle. How many solutions do we have ?

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2 It is given the function $f : (\mathbb{R} - \{0\}) \times (\mathbb{R} - \{0\}) \to \mathbb{R}$ such that $f(a,b) = \left|\frac{|b-a|}{|ab|} + \frac{b+a}{ab} - 1\right| + \frac{|b-a|}{|ab|} + \frac{b+a}{ab} + 1$ where $a, b \neq 0$. Prove that:

$$f(a,b) = 4 \cdot \text{max}\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{2}\right\}$$

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3 If a, b, c are real positive numbers prove that the inequality holds:

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2} + \frac{\sqrt{b^3+c^3}}{b^2+c^2} + \frac{\sqrt{c^3+a^3}}{c^2+a^2} \geq \frac{6(ab+bc+ac)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}$$

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4 It is given a convex hexagon $A_1A_2\cdots A_6$ such that all its interior angles are same valued (congruent). Denote by $a_1 = \overline{A_1A_2},\ a_2 = \overline{A_2A_3},\ \cdots, a_6 = \overline{A_6A_1}$.

$a)$ Prove that holds: $a_1 - a_4 = a_2 - a_5 = a_3 - a_6$ $b)$ Prove that if $a_1, a_2, a_3, ..., a_6$ satisfy the above equation, we can construct a convex hexagon with its same-valued (congruent) interior angles.

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5 Let $n > 1$ be an integer and S_n the set of all permutations $\pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ where π is bijective function. For every permutation $\pi \in S_n$ we define:

$$F(\pi) = \sum_{k=1}^{n} |k - \pi(k)| \text{ and } M_n = \frac{1}{n!}\sum_{\pi \in S_n} F(\pi)$$

where M_n is taken with all permutations $\pi \in S_n$. Calculate the sum M_n.

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