Kosovo National Mathematical Olympiad 2013

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– Grade level 9

1 Prove that:

$\sqrt{10+\sqrt{24}+\sqrt{40}+\sqrt{60}}=\sqrt{2}+\sqrt{3}+\sqrt{5}$

2 Find all integer n such that $n-5$ divide n^2+n-27.

3 For all real numbers a prove that $3(a^4+a^2+1)\geq(a^2+a+1)^2$

4 Find all value of parameter a such that equations $x^2-ax+1=0$ and $x^2-x+a=0$ have at least one same solution.
For this value a find same solution of this equations(real or imaginary).

5 Let ABC be an equilateral triangle, with sidelength equal to a. Let P be a point in the interior of triangle ABC, and let D,E and F be the feet of the altitudes from P on AB,BC and CA, respectively. Prove that $\frac{|PD|+|PE|+|PF|}{3a}=\frac{\sqrt{3}}{6}$

– Grade level 10

1 Let be a,b real numbers such that $|a|\neq|b|$ and $\frac{a+b}{a-b}+\frac{a-b}{a+b}=6$.

Calculate:

$\frac{a^3+b^3}{a^3-b^3}+\frac{a^3-b^3}{a^3+b^3}$

2 Three numbers have sum k (where $k\in\mathbb{R}$) such that the numbers are arethmetic progression.If First of two numbers remain the same and to the third number we add $\frac{k}{6}$ than we have geometry progression.
Find those numbers?

3 How many positive integers which are less or equal with 2013 such that 3 or 5 divide the number.

4 Let be n positive integer than calculate:

$1\cdot 1!+2\cdot 2!+...+n\cdot n!$

5 Let P be a point inside or outside (but not on) of a triangle ABC. Prove that $PA+PB+PC$ is greater than half of the perimeter of the triangle

– Grade level 11

1 Let be z_1 and z_2 two complex numbers such that $|z_1+2z_2|=|2z_1+z_2|$.Prove that for all real numbers a is true $|z_1+az_2|=|az_1+z_2|$

2 Solve equation $27\cdot 3^{3\sin x}=9^{\cos^2 x}$ where $x\in[0,2\pi)$

3 Prove that solution of equation $y=x^2+ax+b$ and $x=y^2+cy+d$ it belong a circle.

4 Let be a,b,c three positive integer.Prove that 4 divide $a^2+b^2+c^2$ only and only if a,b,c are even.

5 Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. . Assume that $ABCD$ has been inscribed in the circle with center O. Prove that AOC separates $ABCD$ into two quadrilaterals of equal area

– Grade level 12

1 Which number is bigger $\sqrt[2012]{2012!}$ or $\sqrt[2013]{2013!}$.

2 Math teacher wrote in a table a polynomial $P(x)$ with integer coefficients and he said:
"Today my daughter have a birthday.If in polynomial $P(x)$ we have $x=a$ where a is the age of my daughter we have $P(a)=a$ and $P(0)=p$ where p is a prime number such that $p>a$."
How old is the daughter of math teacher?

3 Find all numbers x such that:

$$1+2\cdot 2^x+3\cdot 3^x<6^x$$

4 Calculate: $\sqrt{3\sqrt{5\sqrt{3\sqrt{5...}}}}$

5 A trapezium has parallel sides of length equal to a and b ($a<b$), and the distance between the parallel sides is the altitude h. The extensions of the non-parallel lines intersect at a point that is a vertex of two triangles that have as sides the parallel sides of the trapezium. Express the areas of the triangles as functions of a,b and h.