Final Round - Switzerland 2010

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by Martin N.

1 Three coins lie on integer points on the number line. A move consists of choosing and moving two coins, the first one 1 unit to the right and the second one 1 unit to the left.
Under which initial conditions is it possible to move all coins to one single point?

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2 Let $\triangle ABC$ be a triangle with $AB \neq AC$. The incircle with centre I touches BC, CA, AB at D, E, F, respectively. Furthermore let M the midpoint of EF and AD intersect the incircle at $P \neq D$.
Show that $PMID$ ist cyclic.

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3 For $n \in \mathbb{N}$, determine the number of natural solutions (a, b) such that

$$(4a - b)(4b - a) = 2010^n$$

holds.

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4 Let x, y, $z \in \mathbb{R}^+$ satisfying $xyz = 1$. Prove that

$$\frac{(x+y-1)^2}{z} + \frac{(y+z-1)^2}{x} + \frac{(z+x-1)^2}{y} \geqslant x + y + z.$$

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5 Some sides and diagonals of a regular n-gon form a connected path that visits each vertex exactly once. A *parallel pair* of edges is a pair of two different parallel edges of the path. Prove that
(a) if n is even, there is at least one *parallel pair*.
(b) if n is odd, there can't be one single *parallel pair*.

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6 Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(f(x)) + f(f(y)) = 2y + f(x - y)$$

holds.

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7 Let m, n be natural numbers such that $m + n + 1$ is prime and divides $2(m^2 + n^2) - 1$.
Prove that $m = n$.

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8 In a village with at least one inhabitant, there are several associations. Each inhabitant is a member of at least k associations, and any two associations have at most one common member.
Prove that at least k associations have the same number of members.

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9 Let k and k' two concentric circles centered at O, with k' being larger than k. A line through O intersects k at A and k' at B such that O seperates A and B. Another line through O intersects k at E and k' at F such that E separates O and F.
Show that the circumcircle of $\triangle OAE$ and the circles with diametres AB and EF have a common point.

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10 Let $n \geqslant 3$ and P a convex n-gon. Show that P can be, by $n-3$ non-intersecting diagonals, partitioned in triangles such that the circumcircle of each triangle contains the whole area of P. Under which conditions is there exactly one such triangulation?
