Final Round - Switzerland 2011

www.artofproblemsolving.com/community/c4097

by Martin N.

1 At a party, there are 2011 people with a glass of fruit juice each sitting around a circular table. Once a second, they clink glasses obeying the following two rules:
(a) They do not clink glasses crosswise.
(b) At each point of time, everyone can clink glasses with at most one other person.
How many seconds pass at least until everyone clinked glasses with everybody else?

(Swiss Mathematical Olympiad 2011, Final round, problem 1)

---

2 Let $\triangle ABC$ be an acute-angled triangle and let D, E, F be points on BC, CA, AB, respectively, such that

$$\angle AFE = \angle BFD, \quad \angle BDF = \angle CDE \quad \text{and} \quad \angle CED = \angle AEF.$$

Prove that D, E and F are the feet of the perpendiculars through A, B and C on BC, CA and AB, respectively.

(Swiss Mathematical Olympiad 2011, Final round, problem 2)

---

3 For positive integers m and n, find the smalles possible value of $|2011^m - 45^n|$.

(Swiss Mathematical Olympiad, Final round, problem 3)

---

4 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any real numbers $a, b, c, d > 0$ satisfying $abcd = 1$,

$$(f(a) + f(b))(f(c) + f(d)) = (a + b)(c + d)$$

holds true.

(Swiss Mathematical Olympiad 2011, Final round, problem 4)

---

5 Let $\triangle ABC$ be a triangle with circumcircle τ. The tangentlines to τ through A and B intersect at T. The circle through A, B and T intersects BC and AC again at D and E, respectively; CT and BE intersect at F.
Suppose D is the midpoint of BC. Calculate the ratio $BF : BE$.

(Swiss Mathematical Olympiad 2011, Final round, problem 5)

---

6 Let a, b, c, d be positive real numbers satisfying $a + b + c + d = 1$. Show that

$$\frac{2}{(a+b)(c+d)} \leq \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{cd}}.$$

(Swiss Mathematical Olympiad 2011, Final round, problem 6)

---

7 For a given rational number r, find all integers z such that

$$2^z + 2 = r^2.$$

(Swiss Mathematical Olympiad 2011, Final round, problem 7)

---

8 Let $ABCD$ be a parallelogram and H the Orthocentre of $\triangle ABC$. The line parallel to AB through H intersects BC at P and AD at Q while the line parallel to BC through H intersects AB at R and CD at S. Show that P, Q, R and S are concyclic.

(Swiss Mathematical Olympiad 2011, Final round, problem 8)

---

9 For any positive integer n let $f(n)$ be the number of divisors of n ending with 1 or 9 in base 10 and let $g(n)$ be the number of divisors of n ending with digit 3 or 7 in base 10. Prove that $f(n) \geqslant g(n)$ for all nonnegative integers n.

(Swiss Mathematical Olympiad 2011, Final round, problem 9)

---

10 On each square of an $n \times n$-chessboard, there are two bugs. In a move, each bug moves to a (vertically of horizontally) adjacent square. Bugs from the same square always move to different squares. Determine the maximal number of free squares that can occur after one move.

(Swiss Mathematical Olympiad 2011, Final round, problem 10)
