Kyrgyzstan National Olympiad 2010

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1 Given that $a, b, c > 0$ and $a + b + c = 1$. Prove that $\sqrt{\frac{ab}{ab+c}} + \sqrt{\frac{bc}{bc+a}} + \sqrt{\frac{ca}{ca+b}} \leqslant \frac{3}{2}$.

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2 Fifteen pairwise coprime positive integers chosen so that each of them less than 2010. Show that at least one of them is prime.

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3 At the meeting, each person is familiar with 22 people. If two persons A and B know each with one another, among the remaining people they do not have a common friend. For each pair individuals A and B are not familiar with each other, there are among the remaining six common acquaintances. How many people were at the meeting?

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4 Point O is chosen in a triangle ABC such that d_a, d_b, d_c are distance from point O to sides BC, CA, AB, respectively. Find position of point O so that product $d_a \cdot d_b \cdot d_c$ becomes maximum.

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5 Let k be a constant number larger than 1. Find all polynomials $P(x)$ such that $P(x^k) = (P(x))^k$ for all real x.

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6 Let p - a prime, where $p > 11$. Prove that there exists a number k such that the product $p \cdot k$ can be written in the decimal system with only ones.

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7 Find all natural triples (a, b, c), such that: $a-)\, a \le b \le c$ $b-)\, (a, b, c) = 1$ $c-)\, a^2b \mid a^3 + b^3 + c^3,\ b^2c \mid a^3 + b^3 + c^3,\ c^2a \mid a^3 + b^3 + c^3$.

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8 Solve in none-negative integers $x^3 + 7x^2 + 35x + 27 = y^3$.
