Kyrgyzstan National Olympiad 2012

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1 Prove that n must be prime in order to have only one solution to the equation $\frac{1}{x}-\frac{1}{y}=\frac{1}{n}$, $x,y\in\mathbb{N}$.

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2 Given positive real numbers $a_1, a_2, ..., a_n$ with $a_1+a_2+...+a_n=1$. Prove that $\left(\frac{1}{a_1^2}-1\right)\left(\frac{1}{a_2^2}-1\right)...\left(\frac{1}{a_n^2}-1\right)$ $(n^2-1)^n$.

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3 Prove that if the diagonals of a convex quadrilateral are perpendicular, then the feet of perpendiculars dropped from the intersection point of diagonals on the sides of this quadrilateral lie on one circle. Is the converse true?

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4 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(f(x)^2+f(y))=xf(x)+y$,$\forall x,y\in R$.

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5 The sequence of natural numbers is defined as follows: for any $k\geq 1$,$a_{k+2}=a_{k+1}\cdot a_k+1$. Prove that for $k\geq 9$ the number a_k-22 is composite.

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6 The numbers $1,2,\ldots,50$ are written on a blackboard. Each minute any two numbers are erased and their positive difference is written instead. At the end one number remains. Which values can take this number?
