Morocco National Olympiad 2011

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by momo1729, Potla, Agung, mavropnevma

– Grade 11 - First Exam

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1 Solve the following equation in $\mathbb{R}^+$:

$$\begin{cases} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2010 \\ x+y+z=\frac{3}{670} \end{cases}$$

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2 Solve in $\mathbb{R}$ the equation : $(x+1)^5 + (x+1)^4(x-1) + (x+1)^3(x-1)^2 + (x+1)^2(x-1)^3 + (x+1)(x-1)^4 + (x-1)^5 = 0$.

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3 Let a and b be two real numbers and let$M(a,b)=\max\left\{3a^2+2b; 3b^2+2a\right\}$. Find the values of a and b for which $M(a,b)$ is minimal.

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4 Let ABC be a triangle. F and L are two points on the side $[AC]$ such that $AF=LC<AC/2$. Find the mesure of the angle $\angle FBL$ knowing that $AB^2+BC^2=AL^2+LC^2$.

– Grade 11 - Second Exam

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1 Prove that

$$2010<\frac{2^2+1}{2^2-1}+\frac{3^2+1}{3^2-1}+...+\frac{2010^2+1}{2010^2-1}<2010+\frac{1}{2}.$$

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2 Compute the sum

$$S=1+2+3-4-5+6+7+8-9-10+\dots-2010$$

where every three consecutive $+$ are followed by two $-$.

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3 Solve in $\mathbb{R}^3$ the following system

$$\begin{cases} \sqrt{x^2-y}=z-1 \\ \sqrt{y^2-z}=x-1 \\ \sqrt{z^2-x}=y-1 \end{cases}$$

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4 Let ABC be a triangle with area 1 and P the middle of the side $[BC]$. M and N are two points of $[AB]-\left\{A,B\right\}$ and $[AC]-\left\{A,C\right\}$ respectively such that $AM=2MB$ and$CN=2AN$. The two lines (AP) and (MN) intersect in a point D. Find the area of the triangle ADN.

– Grade 11 - Third Exam

1 Find the maximum value of the real constant C such that $x^2+y^2+1\geq C(x+y)$, and $x^2+y^2+xy+1\geq C(x+y)$ for all reals x,y.

2 Prove that the equation $x^2+p|x|=qx-1$ has 4 distinct real solutions if and only if $p+|q|+2<0$ (p and q are two real parameters).

3 Find all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ which verify the relation

$$(x-2)f(y)+f(y+2f(x))=f(x+yf(x)),\qquad\forall x,y\in\mathbb{R}.$$

4 Let $ABCD$ be a convex quadrilateral with angles $\angle ABC$ and $\angle BCD$ not less than 120°. Prove that

$$AC+BD>AB+BC+CD$$

– Grade 11 - Fourth Exam

1 Find all positive integers n such that : $-2^0+2^1-2^2+2^3-2^4+...-(-2)^n=4^0+4^1+4^2+...+4^{2010}$

2 One integer was removed from the set $S=\{1,2,3,...,n\}$ of the integers from 1 to n. The arithmetic mean of the other integers of S is equal to $\frac{163}{4}$.
What integer was removed ?

3 When dividing an integer m by a positive integer n, $(0<n\leq 100)$, a student finds $\frac{m}{n}=0,167a_1a_2....$
Prove that the student made a mistake while computing.

4 (C) and (C') are two circles which intersect in A and B. (D) is a line that moves and passes through A, intersecting (C) in P and (C') in P'.
Prove that the bisector of $[PP']$ passes through a non-moving point.

– Grade 11 - Fifth Exam

1 Given positive reals a,b,c; show that we have

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)\geq 8.$$

2 Let α, β, γ be the angles of a triangle ABC of perimeter $2p$ and R is the radius of its circumscribed circle. (a) Prove that

$$\cot^2\alpha + \cot^2\beta + \cot^2\gamma \geq 3\left(9 \cdot \frac{R^2}{p^2} - 1\right).$$

(b) When do we have equality?

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3 Two circles are tangent to each other internally at a point T. Let the chord AB of the larger circle be tangent to the smaller circle at a point P. Prove that the line TP bisects $\angle ATB$.

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4 The diagonals of a trapezoid $ABCD$ whose bases are $[AB]$ and $[CD]$ intersect at P. Prove that

$$S_{PAB} + S_{PCD} > S_{PBC} + S_{PDA},$$

Where S_{XYZ} denotes the area of $\triangle XYZ$.

– Grade 12 - First Exam

Day 1

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1 Let a and b be two positive real numbers such that $a + b = ab$.
Prove that $\frac{a}{b^2+4} + \frac{b}{a^2+4} \geq \frac{1}{2}$.

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2 Solve in $(\mathbb{R}_+^*)^4$ the following system : $\begin{cases} x + y + z + t = 4 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} = 5 - \frac{1}{xyzt} \end{cases}$

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3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R} \;\; f(x) = max(2xy - f(y))$ where $y \in \mathbb{R}$.

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4 Let ABC be a triangle. The inside bisector of the angle $\angle BAC$ cuts $[BC]$ in L and the circle (C) circumsbribed to the triangle ABC in D. The perpendicular to (AC) going through D cuts $[AC]$ in M and the circle (C) in K. Find the value of $\frac{AM}{MC}$ knowing that $\frac{BL}{LC} = \frac{1}{2}$.

Day 2

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1 Compute the sum

$$S = 1 + 2 + 3 - 4 - 5 + 6 + 7 + 8 - 9 - 10 + \cdots - 2010$$

where every three consecutive $+$ are followed by two $-$.

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2 Let a, b, c be three postive real numbers such that $a + b + c = 1$.
Prove that $9abc \leq ab + ac + bc < 1/4 + 3abc$.

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3 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x,y,\in\mathbb{R}$,

$$xf(x+xy)=xf(x)+f(x^2)\cdot f(y).$$

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4 Let ABC be a triangle with area 1 and P the middle of the side $[BC]$. M and N are two points of $[AB]-\{A,B\}$ and $[AC]-\{A,C\}$ respectively such that $AM=2MB$ and$CN=2AN$. The two lines (AP) and (MN) intersect in a point D. Find the area of the triangle ADN.

Day 3

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1 Find the maximum value of the real constant C such that $x^2+y^2+1\geq C(x+y)$, and $x^2+y^2+xy+1\geq C(x+y)$ for all reals x,y.

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2 Prove that the equation $x^2+p|x|=qx-1$ has 4 distinct real solutions if and only if $p+|q|+2<0$ (p and q are two real parameters).

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3 Find all functions $f:\mathbb{R}\to\mathbb{R}$ which verify the relation

$$(x-2)f(y)+f(y+2f(x))=f(x+yf(x)),\qquad \forall x,y\in\mathbb{R}.$$

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4 Let ABC be a triangle and I the center of its incircle. P is a point inside ABC such that $\angle PBA+\angle PCA=\angle PBC+\angle PCB$. Prove that $AP\geq AI$ with equality iff $P=I$.

Day 4

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1 Find all positive integers n such that : $-2^0+2^1-2^2+2^3-2^4+...-(-2)^n=4^0+4^1+4^2+...+4^{2010}$

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2 One integer was removed from the set $S=\{1,2,3,...,n\}$ of the integers from 1 to n. The arithmetic mean of the other integers of S is equal to $\frac{163}{4}$.
What integer was removed ?

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3 Problem 3 (MAR CP 1992) :
From the digits $1,2,...,9$, we write all the numbers formed by these nine digits (the nine digits are all distinct), and we order them in increasing order as follows : 123456789, 123456798, ..., 987654321. What is the $100000th$ number ?

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4 Two circles C_1 and C_2 intersect in A and B. A line passing through B intersects C_1 in C and C_2 in D. Another line passing through B intersects C_1 in E and C_2 in F, (CF) intersects C_1 and C_2 in P and Q respectively. Make sure that in your diagram, $B,E,C,A,P\in C_1$ and $B,D,F,A,Q\in$

C_2 in this order. Let M and N be the middles of the arcs BP and BQ respectively. Prove that if $CD = EF$, then the points C, F, M, N are cocylic in this order.

Day 5

1 Let x, y, and z be three real positive numbers such that $x^2 + y^2 + z^2 + 2xyz = 1$.
Prove that $2(x + y + z) \leq 3$.

2 Let α, β, γ be the angles of a triangle ABC of perimeter $2p$ and R is the radius of its circumscribed circle. (a) Prove that

$$\cot^2\alpha + \cot^2\beta + \cot^2\gamma \geq 3\left(9\cdot\frac{R^2}{p^2} - 1\right).$$

(b) When do we have equality?

3 Prove that there exist two functions $f, g\colon \mathbb{R} \to \mathbb{R}$, such that $f \circ g$ is strictly decreasing and $g \circ f$ is strictly increasing.

(Poland) Andrzej Komisarski and Marcin Kuczma

4 Let a, b, c, d, m, n be positive integers such that $a^2 + b^2 + c^2 + d^2 = 1989$, $n^2 = max\,\{a, b, c, d\}$ and $a + b + c + d = m^2$.
Find the values of m and n.