APMO 1989

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1 Let $x_1, x_2, \cdots, x_n$ be positive real numbers, and let

$$S = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$(1+x_1)(1+x_2)\cdots(1+x_n) \leq 1 + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \cdots + \frac{S^n}{n!}$$

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2 Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

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3 Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1$, 2, and 3, suppose that B_n is the midpoint of $A_n A_{n+1}$, and suppose that C_n is the midpoint of $A_n B_n$. Suppose that $A_n C_{n+1}$ and $B_n A_{n+2}$ meet at D_n, and that $A_n B_{n+1}$ and $C_n A_{n+2}$ meet at E_n.
Calculate the ratio of the area of triangle $D_1 D_2 D_3$ to the area of triangle $E_1 E_2 E_3$.

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4 Let S be a set consisting of m pairs (a, b) of positive integers with the property that $1 \leq a < b \leq n$. Show that there are at least

$$4m \cdot \frac{(m - \frac{n^2}{4})}{3n}$$

triples (a, b, c) such that (a, b), (a, c), and (b, c) belong to S.

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5 Determine all functions f from the reals to the reals for which

(1) $f(x)$ is strictly increasing and (2) $f(x) + g(x) = 2x$ for all real x,

where $g(x)$ is the composition inverse function to $f(x)$. (Note: f and g are said to be composition inverses if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x.)

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