APMO 1994

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1 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that
(i) For all $x, y \in \mathbb{R}$,

$$f(x) + f(y) + 1 \geq f(x+y) \geq f(x) + f(y)$$

(ii) For all $x \in [0,1)$, $f(0) \geq f(x)$,
(iii) $-f(-1) = f(1) = 1$.

Find all such functions f.

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2 Given a nondegenerate triangle ABC, with circumcentre O, orthocentre H, and circumradius R, prove that $|OH| < 3R$.

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3 Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \leq \sqrt{n}$, then p divides ab. Determine all such n.

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4 Is there an infinite set of points in the plane such that no three points are collinear, and the distance between any two points is rational?

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5 You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:

A	B	C
10	1010	20
100	1100100	400
1000	1111101000	13000
$\vdots$	$\vdots$	$\vdots$

Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.
