**APMO 2003**

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by shobber

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**1**     Let $a, b, c, d, e, f$ be real numbers such that the polynomial

$$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \ldots, 8$. Determine all possible values of $f$.

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**2**     Suppose $ABCD$ is a square piece of cardboard with side length $a$. On a plane are two parallel lines $\ell_1$ and $\ell_2$, which are also $a$ units apart. The square $ABCD$ is placed on the plane so that sides $AB$ and $AD$ intersect $\ell_1$ at $E$ and $F$ respectively. Also, sides $CB$ and $CD$ intersect $\ell_2$ at $G$ and $H$ respectively. Let the perimeters of $\triangle AEF$ and $\triangle CGH$ be $m_1$ and $m_2$ respectively.

Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

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**3**     Let $k \geq 14$ be an integer, and let $p_k$ be the largest prime number which is strictly less than $k$. You may assume that $p_k \geq 3k/4$. Let $n$ be a composite integer. Prove:
(a) if $n = 2p_k$, then $n$ does not divide $(n - k)!$;
(b) if $n > 2p_k$, then $n$ divides $(n - k)!$.

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**4**     Let $a, b, c$ be the sides of a triangle, with $a + b + c = 1$, and let $n \geq 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}.$$

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**5**     Given two positive integers $m$ and $n$, find the smallest positive integer $k$ such that among any $k$ people, either there are $2m$ of them who form $m$ pairs of mutually acquainted people or there are $2n$ of them forming $n$ pairs of mutually unacquainted people.

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