APMO 2006

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by orl

1	Let n be a positive integer. Find the largest nonnegative real number $f(n)$ (depending on n) with the following property: whenever $a_1, a_2, ..., a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n$ is an integer, there exists some i such that $\left\| a_i - \frac{1}{2} \right\| \geq f(n)$.
2	Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden ratio.
3	Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5. Here, we assume that all the checkers are identical.
4	Let A, B be two distinct points on a given circle O and let P be the midpoint of the line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let l be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of l and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.
5	In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colours. Each clown is required to use at least five different colours. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colours and no more than 20 clowns may use any one particular colour. Find the largest number n of clowns so as to make the ringmaster's order possible.