

APMO 2013

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- 1 Let ABC be an acute triangle with altitudes AD , BE , and CF , and let O be the center of its circumcircle. Show that the segments OA , OF , OB , OD , OC , OE dissect the triangle ABC into three pairs of triangles that have equal areas.

- 2 Determine all positive integers n for which $\frac{n^2 + 1}{[\sqrt{n}]^2 + 2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to r .

- 3 For $2k$ real numbers $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$ define a sequence of numbers X_n by

$$X_n = \sum_{i=1}^k [a_i n + b_i] \quad (n = 1, 2, \dots).$$

If the sequence X_N forms an arithmetic progression, show that $\sum_{i=1}^k a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r .

- 4 Let a and b be positive integers, and let A and B be finite sets of integers satisfying
 (i) A and B are disjoint;
 (ii) if an integer i belongs to either to A or to B , then either $i + a$ belongs to A or $i - b$ belongs to B .
 Prove that $a|A| = b|B|$. (Here $|X|$ denotes the number of elements in the set X .)

- 5 Let $ABCD$ be a quadrilateral inscribed in a circle ω , and let P be a point on the extension of AC such that PB and PD are tangent to ω . The tangent at C intersects PD at Q and the line AD at R . Let E be the second point of intersection between AQ and ω . Prove that B, E, R are collinear.