Purple Comet Problems 2003

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1 In eight years Henry will be three times the age that Sally was last year. Twenty five years ago their ages added to 83. How old is Henry now?

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2 What is the smallest number that could be the date of the first Saturday after the second Monday following the second Thursday of a month?

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3 What is the largest integer whose prime factors add to 14?

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4 The lengths of the diagonals of a rhombus are, in inches, two consecutive integers. The area of the rhombus is 210 sq. in. Find its perimeter, in inches.

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5 Let a, b, and c be nonzero real numbers such that $a+\frac{1}{b}=5$, $b+\frac{1}{c}=12$, and $c+\frac{1}{a}=13$. Find $abc+\frac{1}{abc}$.

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6 Evaluate:

$$\frac{1}{\log_2(\frac{1}{6})}-\frac{1}{\log_3(\frac{1}{6})}-\frac{1}{\log_4(\frac{1}{6})}$$

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7 Find the smallest n such that every subset of $\{1,2,3,...,2004\}$ with n elements contains at least two elements that are relatively prime.

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8 Let $ABCDEFGHIJKL$ be a regular dodecagon. Find $\frac{AB}{AF}+\frac{AF}{AB}$.

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9 Let f be a real-valued function of real and positive argument such that $f(x)+3xf(\frac{1}{x})=2(x+1)$ for all real numbers $x>0$. Find $f(2003)$.

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10 How many gallons of a solution which is 15% alcohol do we have to mix with a solution that is 35% alcohol to make 250 gallons of a solution that is 21% alcohol?

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11 If

$$\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+\ldots+20}=\frac{m}{n}$$

where m and n are positive integers with no common divisor, find $m+n$.

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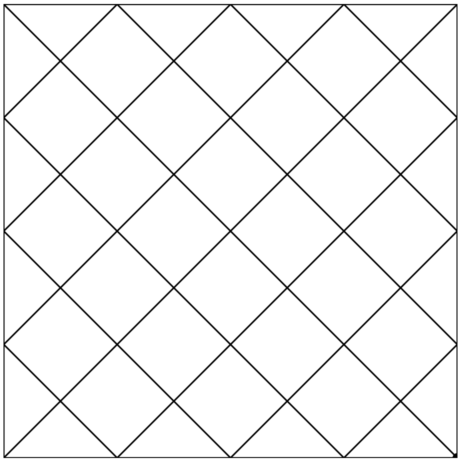
12 How many triangles appear in the diagram below:



13 Let $P(x)$ be a polynomial such that, when divided by $x-2$, the remainder is 3 and, when divided by $x-3$, the remainder is 2. If, when divided by $(x-2)(x-3)$, the remainder is $ax+b$, find a^2+b^2.

14 Let a, b, c be real numbers such that $a^2 - 2 = 3b - c$, $b^2 + 4 = 3 + a$, and $c^2 + 4 = 3a - b$. Find $a^4 + b^4 + c^4$.

15 Let r be a real number such that $\sqrt[3]{r} - \frac{1}{\sqrt[3]{r}} = 2$. Find $r^3 - \frac{1}{r^3}$.

16 Find the largest real number x such that

$$\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2 = \frac{325}{144}.$$

17 Given that $3\sin x + 4\cos x = 5$, where x is in $(0, \frac{\pi}{2})$, find $2\sin x + \cos x + 4\tan x$.

18 A circle radius 320 is tangent to the inside of a circle radius 1000. The smaller circle is tangent to a diameter of the larger circle at a point P. How far is the point P from the outside of the larger circle?

19 Let x_1 and x_2 be the roots of the equation $x^2 + 3x + 1 = 0$. Compute

$$\left(\frac{x_1}{x_2+1}\right)^2 + \left(\frac{x_2}{x_1+1}\right)^2$$

20 In how many ways can we form three teams of four players each from a group of 12 participants?

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21 Let $a_n = \sqrt{1+(1-\frac{1}{n})^2} + \sqrt{1+(1+\frac{1}{n})^2}, n \geq 1$. Evaluate $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{20}}$.

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22 In $\triangle ABC$, max $\{\angle A, \angle B\} = \angle C + 30^\circ$ and $\frac{R}{r} = \sqrt{3}+1$, where R is the radius of the circumcircle and r is the radius of the incircle. Find $\angle C$ in degrees.

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23 For each positive integer m and n define function $f(m,n)$ by $f(1,1) = 1$, $f(m+1,n) = f(m,n)+m$ and $f(m,n+1) = f(m,n) - n$. Find the sum of all the values of p such that $f(p,q) = 2004$ for some q.

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24 In $\triangle ABC$, $\angle A = 30^\circ$ and $AB = AC = 16$ in. Let D lie on segment BC such that $\frac{DB}{DC} = \frac{2}{3}$. Let E and F be the orthogonal projections of D onto AB and AC, respectively. Find $DE + DF$ in inches.

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25 Given that $(1+\tan 1^\circ)(1+\tan 2^\circ)\ldots(1+\tan 45^\circ) = 2^n$, find n.
