www.artofproblemsolving.com/community/c413396
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**–** Team Round

**1** Let $P(x)$, $Q(x)$ be nonconstant polynomials with real number coefficients. Prove that if

$$\lfloor P(y) \rfloor = \lfloor Q(y) \rfloor$$

for all real numbers $y$, then $P(x) = Q(x)$ for all real numbers $x$.

**2** Does there exist a two-variable polynomial $P(x, y)$ with real number coefficients such that $P(x, y)$ is positive exactly when $x$ and $y$ are both positive?

**3** A polyhedron has $7n$ faces. Show that there exist $n + 1$ of the polyhedron's faces that all have the same number of edges.

**4** Let $w = w_1 w_2 \dots w_n$ be a word. Define a *substring* of $w$ to be a word of the form $w_i w_{i+1} \dots w_{j-1} w_j$, for some pair of positive integers $1 \le i \le j \le n$. Show that $w$ has at most $n$ distinct palindromic substrings.

For example, $aaaaa$ has $5$ distinct palindromic substrings, and $abcata$ has $5$ ($a$, $b$, $c$, $t$, $ata$).

**5** Let $ABC$ be an acute triangle. The altitudes $BE$ and $CF$ intersect at the orthocenter $H$, and point $O$ denotes the circumcenter. Point $P$ is chosen so that $\angle APH = \angle OPE = 90^\circ$, and point $Q$ is chosen so that $\angle AQH = \angle OQF = 90^\circ$. Lines $EP$ and $FQ$ meet at point $T$. Prove that points $A$, $T$, $O$ are collinear.

**6** Let $r$ be a positive integer. Show that if a graph $G$ has no cycles of length at most $2r$, then it has at most $|V|^{2016}$ cycles of length exactly $2016r$, where $|V|$ denotes the number of vertices in the graph $G$.

**7** Let $p$ be a prime. A [i]complete residue class modulo $p$[/i] is a set containing at least one element equivalent to $k \pmod p$ for all $k$.
(a) Show that there exists an $n$ such that the $n$th row of Pascal's triangle forms a complete residue class modulo $p$.
(b) Show that there exists an $n \le p^2$ such that the $n$th row of Pascal's triangle forms a complete residue class modulo $p$.

**8** Does there exist an irrational number $\alpha > 1$ such that

$$\lfloor \alpha^n \rfloor \equiv 0 \pmod{2017}$$

for all integers $n \geq 1$?

**9** Let $n$ be an odd positive integer greater than $2$, and consider a regular $n$-gon $\mathcal{G}$ in the plane centered at the origin. Let a *subpolygon* $\mathcal{G}'$ be a polygon with at least $3$ vertices whose vertex set is a subset of that of $\mathcal{G}$. Say $\mathcal{G}'$ is *well-centered* if its centroid is the origin. Also, say $\mathcal{G}'$ is *decomposable* if its vertex set can be written as the disjoint union of regular polygons with at least $3$ vertices. Show that all well-centered subpolygons are decomposable if and only if $n$ has at most two distinct prime divisors.

**10** Let $LBC$ be a fixed triangle with $LB = LC$, and let $A$ be a variable point on arc $LB$ of its circumcircle. Let $I$ be the incenter of $\triangle ABC$ and $\overline{AK}$ the altitude from $A$. The circumcircle of $\triangle IKL$ intersects lines $KA$ and $BC$ again at $U \neq K$ and $V \neq K$. Finally, let $T$ be the projection of $I$ onto line $UV$. Prove that the line through $T$ and the midpoint of $\overline{IK}$ passes through a fixed point as $A$ varies.

**–** Algebra and Number Theory

**1** Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \leq a_i < 3$ for all $0 \leq i \leq n$.

Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

**2** Find the value of

$$\sum_{1 \leq a < b < c} \frac{1}{2^a 3^b 5^c}$$

(i.e. the sum of $\frac{1}{2^a 3^b 5^c}$ over all triples of positive integers $(a, b, c)$ satisfying $a < b < c$)

**3** Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x)f(y) = f(x - y)$. Find all possible values of $f(2017)$.

**4** Find all pairs $(a, b)$ of positive integers such that $a^{2017} + b$ is a multiple of $ab$.

**5** Kelvin the Frog was bored in math class one day, so he wrote all ordered triples $(a, b, c)$ of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all the integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310 \\ a,b,c \in \mathbb{N}}} (a + b + c),$$

where $\mathbb{N}$ denotes the positive integers.

**6** A polynomial $P$ of degree $2015$ satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $\lfloor 2017P(2017) \rfloor$.

**7** Determine the largest real number $c$ such that for any $2017$ real numbers $x_1, x_2, \dots, x_{2017}$, the inequality

$$\sum_{i=1}^{2016} x_i(x_i + x_{i+1}) \geq c \cdot x_{2017}^2$$

holds.

**8** Consider all ordered pairs of integers $(a, b)$ such that $1 \leq a \leq b \leq 100$ and

$$\frac{(a+b)(a+b+1)}{ab}$$

is an integer.

Among these pairs, find the one with largest value of $b$. If multiple pairs have this maximal value of $b$, choose the one with largest $a$. For example choose $(3, 85)$ over $(2, 85)$ over $(4, 84)$. Note that your answer should be an ordered pair.

**9** The Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Find the smallest positive integer $m$ such that $F_m \equiv 0 \pmod{127}$ and $F_{m+1} \equiv 1 \pmod{127}$.

**10** Let $\mathbb{N}$ denote the natural numbers. Compute the number of functions $f : \mathbb{N} \to \{0, 1, \dots, 16\}$ such that

$$f(x+17) = f(x) \qquad \text{and} \qquad f(x^2) \equiv f(x)^2 + 15 \pmod{17}$$

for all integers $x \geq 1$.

**–** Geometry

**1** Let $A$, $B$, $C$, $D$ be four points on a circle in that order. Also, $AB = 3$, $BC = 5$, $CD = 6$, and $DA = 4$. Let diagonals $AC$ and $BD$ intersect at $P$. Compute $\frac{AP}{CP}$.

**2** Let $ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let $\ell$ be a line passing through two sides of triangle $ABC$. Line $\ell$ cuts triangle $ABC$ into two figures, a triangle and a quadrilateral, that have equal perimeter. What is the maximum possible area of the triangle?

**3** Let $S$ be a set of $2017$ points in the plane. Let $R$ be the radius of the smallest circle containing all points in $S$ on either the interior or boundary. Also, let $D$ be the longest distance between two of the points in $S$. Let $a$, $b$ be real numbers such that $a \leq \frac{D}{R} \leq b$ for all possible sets $S$, where $a$ is as large as possible and $b$ is as small as possible. Find the pair $(a, b)$.

**4** Let $ABCD$ be a convex quadrilateral with $AB = 5$, $BC = 6$, $CD = 7$, and $DA = 8$. Let $M$, $P$, $N$, $Q$ be the midpoints of sides $AB$, $BC$, $CD$, $DA$ respectively. Compute $MN^2 - PQ^2$.

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**5** Let $ABCD$ be a quadrilateral with an inscribed circle $\omega$ and let $P$ be the intersection of its diagonals $AC$ and $BD$. Let $R_1$, $R_2$, $R_3$, $R_4$ be the circumradii of triangles $APB$, $BPC$, $CPD$, $DPA$ respectively. If $R_1 = 31$ and $R_2 = 24$ and $R_3 = 12$, find $R_4$.

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**6** In convex quadrilateral $ABCD$ we have $AB = 15$, $BC = 16$, $CD = 12$, $DA = 25$, and $BD = 20$. Let $M$ and $\gamma$ denote the circumcenter and circumcircle of $\triangle ABD$. Line $CB$ meets $\gamma$ again at $F$, line $AF$ meets $MC$ at $G$, and line $GD$ meets $\gamma$ again at $E$. Determine the area of pentagon $ABCDE$.

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**7** Let $\omega$ and $\Gamma$ be circles such that $\omega$ is internally tangent to $\Gamma$ at a point $P$. Let $AB$ be a chord of $\Gamma$ tangent to $\omega$ at a point $Q$. Let $R \neq P$ be the second intersection of line $PQ$ with $\Gamma$. If the radius of $\Gamma$ is $17$, the radius of $\omega$ is $7$, and $\frac{AQ}{BQ} = 3$, find the circumradius of triangle $AQR$.

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**8** Let $ABC$ be a triangle with circumradius $R = 17$ and inradius $r = 7$. Find the maximum possible value of $\sin \frac{A}{2}$.

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**9** Let $ABC$ be a triangle, and let $BCDE$, $CAFG$, $ABHI$ be squares that do not overlap the triangle with centers $X$, $Y$, $Z$ respectively. Given that $AX = 6$, $BY = 7$, and $CA = 8$, find the area of triangle $XYZ$.

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**10** Let $ABCD$ be a quadrilateral with an inscribed circle $\omega$. Let $I$ be the center of $\omega$, and let $IA = 12$, $IB = 16$, $IC = 14$, and $ID = 11$. Let $M$ be the midpoint of segment $AC$. Compute the ratio $\frac{IM}{IN}$, where $N$ is the midpoint of segment $BD$.

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**–** Combinatorics

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**1** Kelvin the Frog is going to roll three fair ten-sided dice with faces labelled $0, 1, \ldots, 9$. First he rolls two dice, and finds the sum of the two rolls. Then he rolls the third die. What is the probability that the sum of the first two rolls equals the third roll?

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**2** How many ways are there to insert $+$'s between the digits of $111111111111111$ (fifteen $1$'s) so that the result will be a multiple of $30$?

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**3** There are $2017$ jars in a row on a table, initially empty. Each day, a nice man picks ten consecutive jars and deposits one coin in each of the ten jars. Later, Kelvin the Frog comes back to see that $N$ of the jars all contain the same positive integer number of coins (i.e. there is an integer $d > 0$ such that $N$ of the jars have exactly $d$ coins). What is the maximum possible value of $N$?

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**4** Sam spends his days walking around the following $2 \times 2$ grid of squares.

| | |
|---|---|
| 1 | 2 |
| 4 | 3 |

Say that two squares are adjacent if they share a side. He starts at the square labeled $1$ and every second walks to an adjacent square. How many paths can Sam take so that the sum of the numbers on every square he visits in his path is equal to $20$ (not counting the square he started on)?

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**5** Kelvin the Frog likes numbers whose digits strictly decrease, but numbers that violate this condition in at most one place are good enough. In other words, if $d_i$ denotes the $i$th digit, then $d_i \le d_{i+1}$ for at most one value of $i$. For example, Kelvin likes the numbers $43210$, $132$, and $3$, but not the numbers $1337$ and $123$. How many $5$-digit numbers does Kelvin like?

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**6** Emily starts with an empty bucket. Every second, she either adds a stone to the bucket or removes a stone from the bucket, each with probability $\frac{1}{2}$. If she wants to remove a stone from the bucket and the bucket is currently empty, she merely does nothing for that second (still with probability $\frac{1}{2}$). What is the probability that after $2017$ seconds her bucket contains exactly $1337$ stones?

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**7** There are $2017$ frogs and $2017$ toads in a room. Each frog is friends with exactly $2$ distinct toads. Let $N$ be the number of ways to pair every frog with a toad who is its friend, so that no toad is paired with more than one frog. Let $D$ be the number of distinct possible values of $N$, and let $S$ be the sum of all possible value of $N$. Find the ordered pair $(D, S)$.

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**8** Kelvin and $15$ other frogs are in a meeting, for a total of $16$ frogs. During the meeting, each pair of distinct frogs becomes friends with probability $\frac{1}{2}$. Kelvin thinks the situation after the meeting is *cool* if for each of the $16$ frogs, the number of friends they made during the meeting is a multiple of $4$. Say that the probability of the situation being cool can be expressed in the form $\frac{a}{b}$, where $a$ and $b$ are relatively prime. Find $a$.

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**9** Let $m$ be a positive integer, and let $T$ denote the set of all subsets of $\{1, 2, \dots, m\}$. Call a subset $S$ of $T$ $\delta$-*good* if for all $s_1, s_2 \in S$, $s_1 \neq s_2$, $|\Delta(s_1, s_2)| \ge \delta m$, where $\Delta$ denotes the symmetric difference (the symmetric difference of two sets is the set of elements that is in exactly one of the two sets). Find the largest possible integer $s$ such that there exists an integer $m$ and $\frac{1024}{2047}$-good set of size $s$.

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**10** Compute the number of possible words $w = w_1 w_2 \dots w_{100}$ satisfying: • $w$ has exactly $50$ $A$'s and $50$ $B$'s (and no other letter). • For $i = 1, 2, \dots, 100$, the number of $A$'s among $w_1, w_2, \dots, w_i$ is at most the number of $B$'s among $w_1, w_2, \dots, w_i$. • For all $i = 44, 45, \dots, 57$, if $w_i$ is a $B$, then $w_{i+1}$ must be a $B$.

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**–** November Theme

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**1** **T**wo ordered pairs $(a, b)$ and $(c, d)$, where $a, b, c, d$ are real numbers, form a basis of the coordinate plane if $ad \neq bc$. Determine the number of ordered quadruples $(a, b, c)$ of integers between $1$ and $3$ inclusive for which $(a, b)$ and $(c, d)$ form a basis for the coordinate plane.

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**2** **H**orizontal parallel segments $AB = 10$ and $CD = 15$ are the bases of trapezoid $ABCD$. Circle $\gamma$ of radius $6$ has center within the trapezoid and is tangent to sides $AB$, $BC$, and $DA$. If side $CD$ cuts out an arc of $\gamma$ measuring $120°$, find the area of $ABCD$.

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**3** **E**milia wishes to create a basic solution with 7

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**4** **M**ary has a sequence $m_2, m_3, m_4, \dots$ , such that for each $b \geq 2$, $m_b$ is the least positive integer m for
which none of the base-$b$ logarithms $log_b(m), log_b(m+1), \dots, log_b(m+2017)$ are integers. Find the largest number in her sequence.

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**5** **E**ach of the integers $1, 2, \dots, 729$ is written in its base-$3$ representation without leading zeroes. The numbers are then joined together in that order to form a continuous string of digits: $12101112202122\dots$ How many times in this string does the substring $012$ appear?

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**6** **R**thea, a distant planet, is home to creatures whose DNA consists of two (distinguishable) strands of bases with a fixed orientation. Each base is one of the letters H, M, N, T, and each strand consists of a sequence of five bases, thus forming five pairs. Due to the chemical properties of the bases, each pair must consist of distinct bases. Also, the bases H and M cannot appear next to each other on the same strand; the same is true for N and T. How many possible DNA sequences are there on Rthea?

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**7** **O**n a blackboard a stranger writes the values of $s_7(n)^2$ for $n = 0, 1, \dots, 7^{20} - 1$, where $s_7(n)$ denotes the sum of digits of $n$ in base $7$. Compute the average value of all the numbers on the board.

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**8** **U**ndecillion years ago in a galaxy far, far away, there were four space stations in the three-dimensional space, each pair spaced 1 light year away from each other. Admiral Ackbar wanted to establish a base somewhere in space such that the sum of squares of the distances from the base to each of the stations does not exceed 15 square light years. (The sizes of the space stations and the base are negligible.) Determine the volume, in cubic light years, of the set of all possible locations for the Admirals base.

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**9** **N**ew this year at HMNT: the exciting game of RNG baseball! In RNG baseball, a team of infinitely many people play on a square field, with a base at each vertex; in particular, one of the bases is called the home base. Every turn, a new player stands at home base and chooses a number n uniformly at random from $\{0, 1, 2, 3, 4\}$. Then, the following occurs:
If $n > 0$, then the player and everyone else currently on the field moves (counterclockwise) around

 

the square by n bases. However, if in doing so a player returns to or moves past the home base, he/she leaves the field immediately and the team scores one point.
If $n = 0$ (a strikeout), then the game ends immediately; the team does not score any more points.
What is the expected number of points that a given team will score in this game?

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**10** Denote $\phi = \frac{\sqrt{5}+1}{2}$ and consider the set of all finite binary strings without leading zeroes. Each string $S$ has a base-$\phi$ value $p(S)$. For example, $p(1101) = \phi^3 + \phi^2 + 1$. For any positive integer n, let $f(n)$ be the number of such strings S that satisfy $p(S) = \frac{\phi^{48n}-1}{\phi^{48}-1}$. The sequence of fractions $\frac{f(n+1)}{f(n)}$ approaches a real number $c$ as $n$ goes to infinity. Determine the value of $c$.

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**–** November General

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**1** Find the sum of all positive integers whose largest proper divisor is $55$. (A proper divisor of $n$ is a divisor that is strictly less than $n$.)

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**2** Determine the sum of all distinct real values of $x$ such that $|||\cdots||x|+x|\cdots|+x|+x| = 1$ where there are $2017$ $x$s in the equation.

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**3** Find the number of integers $n$ with $1 \le n \le 2017$ so that $(n-2)(n-0)(n-1)(n-7)$ is an integer
multiple of $1001$.

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**4** Triangle $ABC$ has $AB = 10$, $BC = 17$, and $CA = 21$. Point $P$ lies on the circle with diameter $AB$. What is the greatest possible area of $APC$?

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**5** Given that $a, b, c$ are integers with $abc = 60$, and that complex number $\omega \neq 1$ satisfies $\omega^3 = 1$, find the minimum possible value of $|a + b\omega + c\omega^2|$.

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**6** A positive integer $n$ is *magical* if $\lfloor \sqrt{\lceil \sqrt{n} \rceil} \rfloor = \lceil \sqrt{\lfloor \sqrt{n} \rfloor} \rceil$. Find the number of magical integers between $1$ and $10,000$ inclusive.

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**7** Reimu has a wooden cube. In each step, she creates a new polyhedron from the previous one by cutting off a pyramid from each vertex of the polyhedron along a plane through the trisection point on each adjacent edge that is closer to the vertex. For example, the polyhedron after the first step has six octagonal faces and eight equilateral triangular faces. How many faces are on the polyhedron after the fifth step?

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**8** Marisa has a collection of $2^8 - 1 = 255$ distinct nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. For each step she takes two subsets chosen uniformly at random from the collection, and replaces them with either their union or their intersection, chosen randomly with equal probability. (The collection is allowed to contain repeated sets.) She repeats this process $2^8 - 2 = 254$ times until there is only one set left in the collection. What is the expected size of this set?

**9** Find the minimum value of $\sqrt{58 - 42x} + \sqrt{149 - 140\sqrt{1 - x^2}}$ where $-1 \leq x \leq 1$.

**10** Five equally skilled tennis players named Allen, Bob, Catheryn, David, and Evan play in a round robin tournament, such that each pair of people play exactly once, and there are no ties. In each of the ten games, the two players both have a 50

**–** November Guts

**1** A random number generator will always output $7$. Sam uses this random number generator once. What is the expected value of the output?

**2** Let $A$, $B$, $C$, $D$, $E$, $F$ be $6$ points on a circle in that order. Let $X$ be the intersection of $AD$ and $BE$, $Y$ is the intersection of $AD$ and $CF$, and $Z$ is the intersection of $CF$ and $BE$. $X$ lies on segments $BZ$ and $AY$ and $Y$ lies on segment $CZ$. Given that $AX = 3$, $BX = 2$, $CY = 4$, $DY = 10$, $EZ = 16$, and $FZ = 12$, find the perimeter of triangle $XYZ$.

**3** Find the number of pairs of integers $(x, y)$ such that $x^2 + 2y^2 < 25$.

**4** Find the number of ordered triples of nonnegative integers $(a, b, c)$ that satisfy

$$(ab + 1)(bc + 1)(ca + 1) = 84.$$

**5** Find the number of ordered triples of positive integers $(a, b, c)$ such that

$$6a + 10b + 15c = 3000.$$

**6** Let $ABCD$ be a convex quadrilateral with $AC = 7$ and $BD = 17$. Let $M$, $P$, $N$, $Q$ be the midpoints of sides $AB$, $BC$, $CD$, $DA$ respectively. Compute $MN^2 + PQ^2$

The official problem statement does not have the final period.

**7** An ordered pair of sets $(A, B)$ is *good* if $A$ is not a subset of $B$ and $B$ is not a subset of $A$. How many ordered pairs of subsets of $\{1, 2, \dots, 2017\}$ are good?

**8** You have $128$ teams in a single elimination tournament. The Engineers and the Crimson are two of these teams. Each of the $128$ teams in the tournament is equally strong, so during each match, each team has an equal probability of winning.

Now, the $128$ teams are randomly put into the bracket.

What is the probability that the Engineers play the Crimson sometime during the tournament?

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**9** Jeffrey writes the numbers $1$ and $100000000 = 10^8$ on the blackboard. Every minute, if $x, y$ are on the board, Jeffery replaces them with

$$\frac{x+y}{2} \quad \text{and} \quad 2\left(\frac{1}{x}+\frac{1}{y}\right)^{-1}.$$

After $2017$ minutes the two numbers are $a$ and $b$. Find $\min(a, b)$ to the nearest integer.

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**10** Let $ABC$ be a triangle in the plane with $AB = 13$, $BC = 14$, $AC = 15$. Let $M_n$ denote the smallest possible value of $(AP^n + BP^n + CP^n)^{\frac{1}{n}}$ over all points $P$ in the plane. Find $\lim_{n\to\infty} M_n$.

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**11** Consider the graph in $3$-space of

$$0 = xyz(x+y)(y+z)(z+x)(x-y)(y-z)(z-x).$$

This graph divides $3$-space into $N$ connected regions. What is $N$?

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**12** In a certain college containing $1000$ students, students may choose to major in exactly one of math, computer science, finance, or English. The *diversity ratio* $d(s)$ of a student $s$ is the defined as number of students in a different major from $s$ divided by the number of students in the same major as $s$ (including $s$). The *diversity* $D$ of the college is the sum of all the diversity ratios $d(s)$.

Determine all possible values of $D$.

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**13** The game of Penta is played with teams of five players each, and there are five roles the players can play. Each of the five players chooses two of five roles they wish to play. If each player chooses their roles randomly, what is the probability that each role will have exactly two players?

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**14** Mrs. Toad has a class of $2017$ students, with unhappiness levels $1, 2, \ldots, 2017$ respectively. Today in class, there is a group project and Mrs. Toad wants to split the class in exactly $15$ groups. The unhappiness level of a group is the average unhappiness of its members, and the unhappiness of the class is the sum of the unhappiness of all $15$ groups. What's the minimum unhappiness of the class Mrs. Toad can achieve by splitting the class into $15$ groups?

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**15** Start by writing the integers $1, 2, 4, 6$ on the blackboard. At each step, write the smallest positive integer $n$ that satisfies both of the following properties on the board.

 

- $n$ is larger than any integer on the board currently.
- $n$ cannot be written as the sum of $2$ distinct integers on the board.

Find the $100$-th integer that you write on the board. Recall that at the beginning, there are already $4$ integers on the board.

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**16** Let $a$ and $b$ be complex numbers satisfying the two equations

$$a^3 - 3ab^2 = 36$$
$$b^3 - 3ba^2 = 28i.$$

Let $M$ be the maximum possible magnitude of $a$. Find all $a$ such that $|a| = M$.

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**17** Sean is a biologist, and is looking at a strng of length $66$ composed of the letters $A$, $T$, $C$, $G$. A *substring* of a string is a contiguous sequence of letters in the string. For example, the string $AGTC$ has $10$ substrings: $A$, $G$, $T$, $C$, $AG$, $GT$, $TC$, $AGT$, $GTC$, $AGTC$. What is the maximum number of distinct substrings of the string Sean is looking at?

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**18** Let $ABCD$ be a quadrilateral with side lengths $AB = 2$, $BC = 3$, $CD = 5$, and $DA = 4$. What is the maximum possible radius of a circle inscribed in quadrilateral $ABCD$?

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**19** Find (in terms of $n \geq 1$) the number of terms with odd coefficients after expanding the product:

$$\prod_{1 \leq i < j \leq n} (x_i + x_j)$$

e.g., for $n = 3$ the expanded product is given by $x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_2^2x_1 + x_3^2x_1 + x_3^2x_2 + 2x_1x_2x_3$ and so the answer would be $6$.

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**20** For positive integers $a$ and $N$, let $r(a, N) \in \{0, 1, \dots, N-1\}$ denote the remainder of $a$ when divided by $N$. Determine the number of positive integers $n \leq 1000000$ for which

$$r(n, 1000) > r(n, 1001).$$

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**21** Let $P$ and $A$ denote the perimeter and area respectively of a right triangle with relatively prime integer side-lengths. Find the largest possible integral value of $\frac{P^2}{A}$

The official statement does not have the final period.

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**22** Kelvin the Frog and $10$ of his relatives are at a party. Every pair of frogs is either *friendly* or *unfriendly*. When $3$ pairwise friendly frogs meet up, they will gossip about one another and end up in a *fight* (but stay *friendly* anyway). When $3$ pairwise unfriendly frogs meet up, they will also end up in a *fight*. In all other cases, common ground is found and there is no fight. If all $\binom{11}{3}$ triples of frogs meet up exactly once, what is the minimum possible number of fights?

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**23** Five points are chosen uniformly at random on a segment of length $1$. What is the expected distance between the closest pair of points?

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**24** At a recent math contest, Evan was asked to find $2^{2016} \pmod{p}$ for a given prime number $p$ with $100 < p < 500$. Evan has forgotten what the prime $p$ was, but still remembers how he solved it:

- Evan first tried taking $2016$ modulo $p-1$, but got a value $e$ larger than $100$.
- However, Evan noted that $e - \frac{1}{2}(p-1) = 21$, and then realized the answer was $-2^{21} \pmod{p}$.

What was the prime $p$?

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**25** Find all real numbers $x$ satisfying the equation $x^3 - 8 = 16\sqrt[3]{x+1}$.

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**26** Kelvin the Frog is hopping on a number line (extending to infinity in both directions). Kelvin starts at $0$. Every minute, he has a $\frac{1}{3}$ chance of moving $1$ unit left, a $\frac{1}{3}$ chance of moving $1$ unit right, and $\frac{1}{3}$ chance of getting eaten. Find the expected number of times Kelvin returns to $0$ (not including the start) before he gets eaten.

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**27** Find the smallest possible value of $x+y$ where $x, y \geq 1$ and $x$ and $y$ are integers that satisfy $x^2 - 29y^2 = 1$.

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**28** Let $\ldots, a_{-1}, a_0, a_1, a_2, \ldots$ be a sequence of positive integers satisfying the folloring relations: $a_n = 0$ for $n < 0$, $a_0 = 1$, and for $n \geq 1$,

$$a_n = a_{n-1} + 2(n-1)a_{n-2} + 9(n-1)(n-2)a_{n-3} + 8(n-1)(n-2)(n-3)a_{n-4}.$$

Compute

$$\sum_{n \geq 0} \frac{10^n a_n}{n!}.$$

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**29** Yang has the sequence of integers $1, 2, \ldots, 2017$. He makes $2016$ *swaps* in order, where a swap changes the positions of two integers in the sequence. His goal is to end with $2, 3, \ldots, 2017, 1$. How many different sequences of swaps can Yang do to achieve his goal?

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**30** Consider an equilateral triangular grid $G$ with $20$ points on a side, where each row consists of points spaced $1$ unit apart. More specifically, there is a single point in the first row, two points in the second row, ..., and $20$ points in the last row, for a total of $210$ points. Let $S$ be a closed non-self-intersecting polygon which has $210$ vertices, using each point in $G$ exactly once. Find the sum of all possible values of the area of $S$.

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**31** A baseball league has $6$ teams. To decide the schedule for the league, for each pair of teams, a coin is flipped. If it lands head, they will play a game this season, in which one team wins and one team loses. If it lands tails, they don't play a game that season. Define the *imbalance* of this schedule to be the minimum number of teams that will end up undefeated, i.e. lose $0$ games. Find the expected value of the imbalance in this league.

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**32** Let $a$, $b$, $c$ be non-negative real numbers such that $ab + bc + ca = 3$. Suppose that

$$a^3b + b^3c + c^3a + 2abc(a + b + c) = \frac{9}{2}.$$

What is the maximum possible value of $ab^3 + bc^3 + ca^3$?

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**33** Welcome to the **USAYNO**, where each question has a yes/no answer. Choose any subset of the following six problems to answer. If you answer $n$ problems and get them **all** correct, you will receive $\max(0, (n-1)(n-2))$ points. If any of them are wrong (or you leave them all blank), you will receive $0$ points.

Your answer should be a six-character string containing 'Y' (for yes), 'N' (for no), or 'B' (for blank). For instance if you think 1, 2, and 6 are 'yes' and 3 and 4 are 'no', you should answer YYNNBY (and receive $12$ points if all five answers are correct, 0 points if any are wrong).

(a) $a, b, c, d, A, B, C,$ and $D$ are positive real numbers such that $\frac{a}{b} > \frac{A}{B}$ and $\frac{c}{d} > \frac{C}{D}$. Is it necessarily true that $\frac{a+c}{b+d} > \frac{A+C}{B+D}$?

(b) Do there exist irrational numbers $\alpha$ and $\beta$ such that the sequence $\lfloor\alpha\rfloor+\lfloor\beta\rfloor, \lfloor 2\alpha\rfloor+\lfloor 2\beta\rfloor, \lfloor 3\alpha\rfloor+\lfloor 3\beta\rfloor, \dots$ is arithmetic?

(c) For any set of primes $\mathbb{P}$, let $S_{\mathbb{P}}$ denote the set of integers whose prime divisors all lie in $\mathbb{P}$. For instance $S_{\{2,3\}} = \{2^a3^b \mid a, b \geq 0\} = \{1, 2, 3, 4, 6, 8, 9, 12, \dots\}$. Does there exist a finite set of primes $\mathbb{P}$ and integer polynomials $P$ and $Q$ such that $\gcd(P(x), Q(y)) \in S_{\mathbb{P}}$ for all $x, y$?

(d) A function $f$ is called **P-recursive** if there exists a positive integer $m$ and real polynomials $p_0(n), p_1(n), \dots, p_m(n)$, not all zero, satisfying

$$p_m(n)f(n+m) = p_{m-1}(n)f(n+m-1) + \cdots + p_0(n)f(n)$$

for all $n$. Does there exist a P-recursive function $f$ satisfying $\lim_{n\to\infty} \frac{f(n)}{n^{\sqrt{2}}} = 1$?

(e) Does there exist a **nonpolynomial** function $f : \mathbb{Z} \to \mathbb{Z}$ such that $a - b$ divides $f(a) - f(b)$ for all integers $a \neq b$?

(f) Do there exist periodic functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) + g(x) = x$ for all $x$?

A clarification was issued for problem 33(d) during the test. I have included it above.

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**34** Welcome to the **USAYNO**, where each question has a yes/no answer. Choose any subset of the following six problems to answer. If you answer $n$ problems and get them **all** correct, you will receive $\max(0, (n-1)(n-2))$ points. If any of them are wrong (or you leave them all blank), you will receive $0$ points.

 

Your answer should be a six-character string containing 'Y' (for yes), 'N' (for no), or 'B' (for blank). For instance if you think 1,2, and 6 are 'yes' and 3 and 4 are 'no', you should answer YYNNBY (and receive $12$ points if all five answers are correct, 0 points if any are wrong).

(a) Can $1000$ queens be placed on a $2017 \times 2017$ chessboard such that every square is attacked by some queen? A square is attacked by a queen if it lies on the same row, column, or diagonal as the queen.

(b) A $2017 \times 2017$ grid of squares originally contains a $0$ in each square. At any step, Kelvin the Frog choose two adjacent squares (two squares are adjacent if they share a side) and increments the numbers in both of them by $1$. Can Kelvin make every square contain a different power of $2$?

(c) A *tournament* consists of single games between every pair of players, where each game has a winner and a loser with no ties. A set of people is *dominated* if there exists a player who beats all of them. Does there exist a tournament in which every set of $2017$ people is dominated?

(d) Every cell of a $19 \times 19$ grid is colored either red, yellow, green, or blue. Does there necessarily exist a rectangle whose sides are parallel to the grid, all of whose vertices are the same color?

(e) Does there exist a $c \in \mathbb{R}^+$ such that $\max(|A \cdot A|, |A + A|) \geq c|A| \log^2 |A|$ for all finite sets $A \subset \mathbb{Z}$?

(f) Can the set $\{1, 2, \ldots, 1093\}$ be partitioned into $7$ subsets such that each subset is sum-free (i.e. no subset contains $a, b, c$ with $a + b = c$)?

The USAYNO disclaimer is only included in problem 33. I have included it here for convenience.

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**35** Welcome to the **USAYNO**, where each question has a yes/no answer. Choose any subset of the following six problems to answer. If you answer $n$ problems and get them **all** correct, you will receive $\max(0, (n-1)(n-2))$ points. If any of them are wrong (or you leave them all blank), you will receive $0$ points.

Your answer should be a six-character string containing 'Y' (for yes), 'N' (for no), or 'B' (for blank). For instance if you think 1, 2, and 6 are 'yes' and 3 and 4 are 'no', you should answer YYNNBY (and receive $12$ points if all five answers are correct, 0 points if any are wrong).

(a) Does there exist a finite set of points, not all collinear, such that a line between any two points in the set passes through a third point in the set?

(b) Let $ABC$ be a triangle and $P$ be a point. The *isogonal conjugate* of $P$ is the intersection of the reflection of line $AP$ over the $A$-angle bisector, the reflection of line $BP$ over the $B$-angle bisector, and the reflection of line $CP$ over the $C$-angle bisector. Clearly the incenter is its own isogonal conjugate. Does there exist another point that is its own isogonal conjugate?

(c) Let $F$ be a convex figure in a plane, and let $P$ be the largest pentagon that can be inscribed in $F$. Is it necessarily true that the area of $P$ is at least $\frac{3}{4}$ the area of $F$?

(d) Is it possible to cut an equilateral triangle into $2017$ pieces, and rearrange the pieces into a square?

(e) Let $ABC$ be an acute triangle and $P$ be a point in its interior. Let $D, E, F$ lie on $BC, CA, AB$ respectively so that $PD$ bisects $\angle BPC$, $PE$ bisects $\angle CPA$, and $PF$ bisects $\angle APB$. Is it necessarily true that $AP + BP + CP \geq 2(PD + PE + PF)$?

(f) Let $P_{2018}$ be the surface area of the $2018$-dimensional unit sphere, and let $P_{2017}$ be the surface area of the $2017$-dimensional unit sphere. Is $P_{2018} > P_{2017}$?

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**36** Welcome to the **USAYNO**, where each question has a yes/no answer. Choose any subset of the following six problems to answer. If you answer $n$ problems and get them **all** correct, you will receive $\max(0, (n-1)(n-2))$ points. If any of them are wrong (or you leave them all blank), you will receive $0$ points.

Your answer should be a six-character string containing 'Y' (for yes), 'N' (for no), or 'B' (for blank). For instance if you think 1, 2, and 6 are 'yes' and 3 and 4 are 'no', you should answer YYNNBY (and receive $12$ points if all five answers are correct, 0 points if any are wrong).

(a) Does $\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \pmod{p^2}$ for all odd prime numbers $p$? (Note that $\frac{1}{i}$ denotes the number such that $i \cdot \frac{1}{i} \equiv 1 \pmod{p^2}$)

(b) Do there exist $2017$ positive perfect cubes that sum to a perfect cube?

(c) Does there exist a right triangle with rational side lengths and area $5$?

(d) A *magic square* is a $3 \times 3$ grid of numbers, all of whose rows, columns, and major diagonals sum to the same value. Does there exist a magic square whose entries are all different prime numbers?

(e) Is $\prod_p \frac{p^2+1}{p^2-1} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \ldots$ a rational number?

(f) Do there exist infinite number of pairs of *distinct* integers $(a, b)$ such that $a$ and $b$ have the same set of prime divisors, and $a + 1$ and $b + 1$ have the same set of prime divisors?

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A clarification was issued for problem 36(d) during the test. I have included it above.

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