Purple Comet Problems 2004

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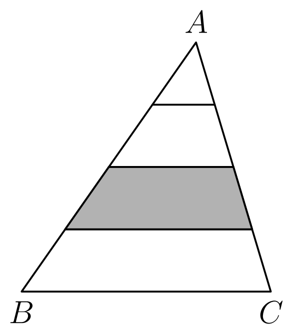
by youarebad, Binomial-theorem

–	Middle School
1	This year February 29 fell on a Sunday. In what year will February 29 next fall on a Sunday?
2	If $h(a,b,c) = \frac{abc}{a+b+c}$, find $h(3\sqrt{5}, 6\sqrt{5}, 9\sqrt{5})$.
3	In $\triangle ABC$, three lines are drawn parallel to side BC dividing the altitude of the triangle into four equal parts. If the area of the second largest part is 35, what is the area of the whole $\triangle ABC$? 
4	If the numbers $2a+2$ and $2b+2$ add up to 2004, find the sum of the numbers $\frac{a}{2}-2$ and $\frac{b}{2}-2$
5	Write the number $2004_{(5)}$ [2004 base 5] as a number in base 6.
6	How many different positive integers divide $10!$?
7	A rectangle has area 1100. If the length is increased by ten percent and the width is decreased by ten percent, what is the area of the new rectangle?
8	The number $2.5081081081081\ldots$ can be written as $\frac{m}{n}$ where m and n are natural numbers with no common factors. Find $m+n$.
9	How many positive integers less that 200 are relatively prime to either 15 or 24?
10	One rainy afternoon you write the number 1 once, the number 2 twice, the number 3 three times, and so forth until you have written the number 99 ninety-nine times. What is the 2005 th digit

that you write?

11 Find the sum of all integers x satisfying $1 + 8x \le 358 - 2x \le 6x + 94$.

12 If $f(x, y) = xy + 2x + y + 1$, find $f(f(2, f(3, 4)), 5)$.

13 How many three digit numbers are made up of three distinct digits?

14 A polygon has five times as many diagonals as it has sides. How many vertices does the polygon have?

15 Find the prime number p for which $p + 2500$ is a perfect square.

16 A week ago, Sandys seasonal Little League batting average was 360. After five more at bats this week, Sandys batting average is up to 400. What is the smallest number of hits that Sandy could have had this season?

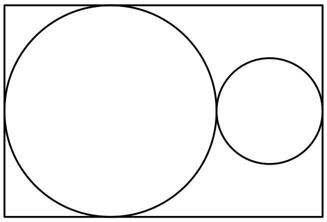
17 A week ago, Sandys seasonal Little League batting average was 360. After five more at bats this week, Sandys batting average is up to 400. What is the smallest number of hits that Sandy could have had this season?

18 Find the number of addition problems in which a two digit number is added to a second two digit number to give a two digit answer, such as in the three examples:

$$\begin{array}{c} 23 \\ \underline{42} \\ 65 \end{array}, \quad \begin{array}{c} 36 \\ \underline{36} \\ 72 \end{array}, \quad \begin{array}{c} 42 \\ \underline{23} \\ 65 \end{array}.$$

19 Find n such that $n - 76$ and $n + 76$ are both cubes of positive integers.

20 A circle with area 40 is tangent to a circle with area 10. Let R be the smallest rectangle containing both circles. The area of R is $\frac{n}{\pi}$. Find n.



21 Find the number of different quadruples (a, b, c, d) of positive integers such that $ab = cd = a + b + c + d - 3$.

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22 Two circles have radii 15 and 95. If the two external tangents to the circles intersect at 60 degrees, how far apart are the centers of the circles?

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23 A cubic block with dimensions n by n by n is made up of a collection of 1 by 1 by 1 unit cubes. What is the smallest value of n so that if the outer layer of unit cubes are removed from the block, more than half the original unit cubes will still remain?

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24 Let a be a real number greater than 1 such that $\frac{20a}{a^2+1} = \sqrt{2}$. Find $\frac{14a}{a^2-1}$.

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25 In the addition problem

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  W H I T E
+ W A T E R
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P I C N I C
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each distinct letter represents a dierent digit. Find the number represented by the answer PICNIC.

– High School

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1 How many different positive integers divide $10!$?

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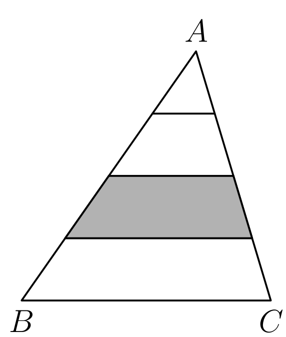
2 In $\triangle ABC$, three lines are drawn parallel to side BC dividing the altitude of the triangle into four equal parts. If the area of the second largest part is 35, what is the area of the whole $\triangle ABC$?



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3 How many real numbers are roots of the polynomial

$$x^9 - 37x^8 - 2x^7 + 74x^6 + x^4 - 37x^3 - 2x^2 + 74x?$$

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4 Find x so that $2^{2^{3^{2^{2}}}} = 4^{4^x}$.

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5 The number $2.5081081081081\ldots$ can be written as m/n where m and n are natural numbers with no common factors. Find $m+n$.

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6 Evaluate the product

$$\left(1+\frac{2}{3}\right)\left(1+\frac{2}{4}\right)\left(1+\frac{2}{5}\right)\cdots\left(1+\frac{2}{98}\right).$$

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7 How many positive integers less that 200 are relatively prime to either 15 or 24?

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8 One rainy afternoon you write the number 1 once, the number 2 twice, the number 3 three times, and so forth until you have written the number 99 ninety-nine times. What is the 2005 th digit that you write?

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9 Let M and m be the largest and the smallest values of x, respectively, which satisfy $4x(x-5) \le 375$. Find $M-m$.

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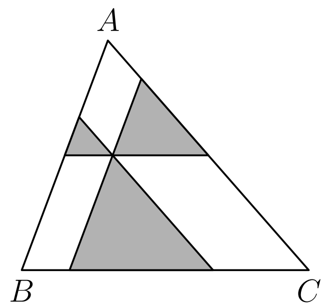
10 Three lines are drawn parallel to each of the three sides of $\triangle ABC$ so that the three lines intersect in the interior of ABC. The resulting three smaller triangles have areas 1, 4, and 9. Find the area of $\triangle ABC$.



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11 How far is it from the point $(9, 17)$ to its reflection across the line

$$3x+4y=15?$$

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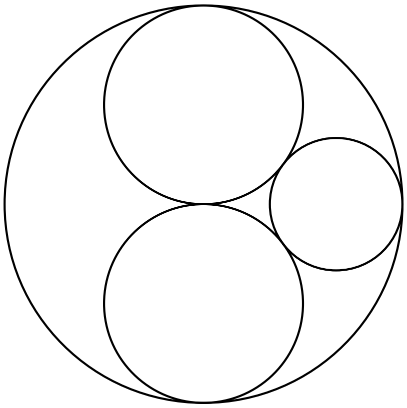
12 The diagram shows a circle with radius 24 which contains two circles with radius 12 tangent to each other and the larger circle. The smallest circle is tangent to the three other circles. What is the radius of the smallest circle?



13 A cubic block with dimensions n by n by n is made up of a collection of 1 by 1 by 1 unit cubes. What is the smallest value of n so that if the outer two layers of unit cubes are removed from the block, more than half the original unit cubes will still remain?

14 Two circles have radii 15 and 95. If the two external tangents to the circles intersect at 60 degrees, how far apart are the centers of the circles?

15 Jerry purchased some stock for $\$14,400$ at the same time that Susan purchased a bond for $\$6,250$. Jerrys investment went up 20 percent the first year, fell 10 percent the second year, and rose another 20 percent the third year. Susans investment grew at a constant rate of compound interest for three years. If both investments are worth the same after three years, what was the annual percentage increase of Susans investment?

16 Find the total length of the set of real numbers satisfying

$$\frac{x^2 - 80x + 1500}{x^2 - 55x + 700} < 0.$$

17 We want to paint some identically-sized cubes so that each face of each cube is painted a solid color and each cube is painted with six different colors. If we have seven different colors to choose from, how many distinguishable cubes can we produce?

18 As x ranges over the interval $(0, \infty)$, the function

$$\sqrt{9x^2 + 173x + 900} - \sqrt{9x^2 + 77x + 900}$$

ranges over the interval $(0, M)$. Find M.

19 There are three bags. One bag contains three green candies and one red candy. One bag contains two green candies and two red candies. One bag contains one green candy and three red candies. A child randomly selects one of the bags, randomly chooses a first candy from that bag, and eats the candy. If the first candy had been green, the child randomly chooses one of the other two bags and randomly selects a second candy from that bag. If the first candy had been red, the child randomly selects a second candy from the same bag as the first candy. If the probability that the second candy is green is given by the fraction m/n in lowest terms, find $m+n$.

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20 A 70 foot pole stands vertically in a plane supported by three 490 foot wires, all attached to the top of the pole, pulled taut, and anchored to three equally spaced points in the plane. How many feet apart are any two of those anchor points?

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21 Define $a_k = (k^2+1)k!$ and $b_k = a_1 + a_2 + a_3 + \cdots + a_k$. Let

$$\frac{a_{100}}{b_{100}} = \frac{m}{n}$$

where m and n are relatively prime natural numbers. Find $n-m$.

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22 How many non-overlapping 2 by 2 squares will fit into a circle with radius 8?

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23 Let a and b be real numbers satisfying

$$a^4 + 8b = 4(a^3 - 1) - 16\sqrt{3}$$

and

$$b^4 + 8a = 4(b^3 - 1) + 16\sqrt{3}.$$

Find $a^4 + b^4$.

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24 The determinant

$$\begin{vmatrix} 3 & -2 & 5 \\ 7 & 1 & -4 \\ 5 & 2 & 3 \end{vmatrix}$$

has the same value as the determinant

$$\begin{vmatrix} x & 1+x & 2+x \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

Find x.

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