Purple Comet Problems 2005

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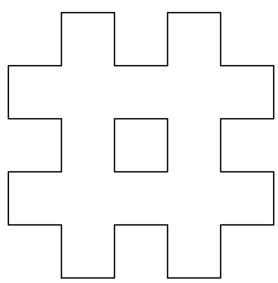
by youarebad

– Middle School

1 A cubic inch of the newly discovered material madelbromium weighs 5 ounces. How many pounds will a cubic yard of madelbromium weigh?

2 Jerry is mowing a rectangular lawn which is 77 feet north to south by 83 feet east to west. His lawn mower cuts a path 18 inches wide. Jerry mows the grass by cutting a path from west to east across the north side of the lawn and then making a right turn cutting a path along the east side of the lawn. When he completes mowing each side of the lawn, he continues by making right turns to mow a path along the next side. How many right turns will he make?

3 Four rectangular strips each measuring 4 by 16 inches are laid out with two vertical strips crossing two horizontal strips forming a single polygon which looks like a tic-tack-toe pattern. What is the perimeter of this polygon?



4 Fill in numbers in the boxes below so that the sum of the entries in each three consecutive boxes is 2005. What is the number that goes into the leftmost box?

		999					888	

5 A palindrome is a number that reads the same forwards and backwards such as 3773 or 42924. Find the sum of the twelve smallest five digit palindromes.

6 We glue together 990 one inch cubes into a 9 by 10 by 11 inch rectangular solid. Then we paint the outside of the solid. How many of the original 990 cubes have just one of their sides painted?

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7 Bills age is one third larger than Tracys age. In 30 years Bills age will be one eighth larger than Tracys age. How many years old is Bill?

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8 The number 1 is special. The number 2 is special because it is relatively prime to 1. The number 3 is not special because it is not relatively prime to the sum of the special numbers less than it, $1+2$. The number 4 is special because it is relatively prime to the sum of the special numbers less than it. So, a number bigger than 1 is special only if it is relatively prime to the sum of the special numbers less than it. Find the twentieth special number.

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9 Find the number of nonnegative integers n for which $(n^2-3n+1)^2+1$ is a prime number

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10 What is the 1000 th digit to the right of the decimal point in the decimal representation of $\frac{37}{5500}$?

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11 The work team was working at a rate fast enough to process 1250 items in ten hours. But after working for six hours, the team was given an additional 150 items to process. By what percent does the team need to increase its rate so that it can still complete its work within the ten hours?

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12 Four mathletes and two coaches sit at a circular table. How many distinct arrangements are there of these six people if the two coaches sit opposite each other?

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13 Find x such that

$$\frac{\frac{5}{x-50}+\frac{7}{x+25}}{\frac{2}{x-50}-\frac{3}{x+25}}=17.$$

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14 Eight identical cubes with of size $1\times1\times1$ each have the numbers 1 through 6 written on their faces with the number 1 written on the face opposite number 2, number 3 written on the face opposite number 5, and number 4 written on the face opposite number 6. The eight cubes are stacked into a single $2\times2\times2$ cube. Add all of the numbers appearing on the outer surface of the new cube. Let M be the maximum possible value for this sum, and N be the minimum possible value for this sum. Find $M-N$.

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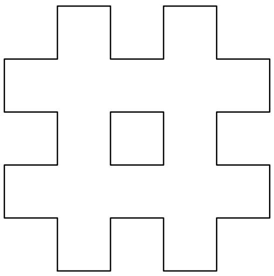
15 And it came to pass that Jeb owned over a thousand chickens. So Jeb counted his chickens. And Jeb reported the count to Hannah. And Hannah reported the count to Joshua. And Joshua reported the count to Caleb. And Caleb reported the count to Rachel. But as fate would have it, Jeb had over-counted his chickens by nine chickens. Then Hannah interchanged the last two digits of the count before reporting it to Joshua. And Joshua interchanged the first and the third digits of the number reported to him before reporting it to Caleb. Then Caleb doubled the number reported to him before reporting it to Rachel. Now it so happens that the count reported to Rachel was the correct number of chickens that Jeb owned. How many chickens was that?

– High School

1 The cost of producing each item is inversely proportional to the square root of the number of items produced. The cost of producing ten items is $\$2100$. If items sell for $\$30$ each, how many items need to be sold so that the producers break even?

2 We glue together 990 one inch cubes into a 9 by 10 by 11 inch rectangular solid. Then we paint the outside of the solid. How many of the original 990 cubes have just one of their sides painted?

3 Four rectangular strips each measuring 4 by 16 inches are laid out with two vertical strips crossing two horizontal strips forming a single polygon which looks like a tic-tack-toe pattern. What is the perimeter of this polygon?



4 A palindrome is a number that reads the same forwards and backwards such as 3773 or 42924. What is the smallest 9 digit palindrome which is a multiple of 3 and has at least two digits which are 5's and two digits which are 7's?

5 In January Jeffs investment went up by three quarters. In February it went down by one quarter. In March it went up by one third. In April it went down by one fifth. In May it went up by one seventh. In June Jeffs investment fell by $\frac{m}{n}$ where m and n are relatively prime positive integers. If Jeffs investment was worth the same amount at the end of June as it had been at the beginning of January, find $m+n$.

6 $ABCDE$ is a regular pentagon. What is the degree measure of the acute angle at the intersection of line segments AC and BD?

7 The graph of the equation $y = 5x + 24$ intersects the graph of the equation $y = x^2$ at two points. The two points are a distance $\sqrt{N}$ apart. Find N.

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8 Find x if

$$\frac{1}{\frac{1}{\frac{1}{\frac{1}{x}+\frac{1}{2}}+\frac{1}{\frac{1}{x}+\frac{1}{2}}}+\frac{1}{\frac{1}{\frac{1}{x}+\frac{1}{2}}+\frac{1}{\frac{1}{x}+\frac{1}{2}}}} = \frac{x}{36}.$$

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9 Let T be a $30-60-90$ triangle with hypotenuse of length 20. Three circles, each externally tangent to the other two, have centers at the three vertices of T. The area of the union of the circles intersected with T is $(m+n\sqrt{3})\pi$ for rational numbers m and n. Find $m+n$.

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10 A jar contains 2 yellow candies, 4 red candies, and 6 blue candies. Candies are randomly drawn out of the jar one-by-one and eaten. The probability that the 2 yellow candies will be eaten before any of the red candies are eaten is given by the fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

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11 The straight river is one and a half kilometers wide and has a current of 8 kilometers per hour. A boat capable of traveling 10 kilometers per hour in still water, sets out across the water. How many minutes will it take the boat to reach a point directly across from where it started?

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12 Find the number of nonnegative integers n for which $(n^2-3n+1)^2+1$ is a prime number.

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13 The work team was working at a rate fast enough to process 1250 items in ten hours. But after working for six hours, the team was given an additional 165 items to process. By what percent does the team need to increase its rate so that it can still complete its work within the ten hours?

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14 Four mathletes and two coaches sit at a circular table. How many distinct arrangements are there of these six people if the two coaches sit opposite each other?

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15 And it came to pass that Jeb owned over a thousand chickens. So Jeb counted his chickens. And Jeb reported the count to Hannah. And Hannah reported the count to Joshua. And Joshua reported the count to Caleb. And Caleb reported the count to Rachel. But as fate would have it, Jeb had over-counted his chickens by nine chickens. Then Hannah interchanged the last two digits of the count before reporting it to Joshua. And Joshua interchanged the first and the third digits of the number reported to him before reporting it to Caleb. Then Caleb doubled the number reported to him before reporting it to Rachel. Now it so happens that the count reported to Rachel was the correct number of chickens that Jeb owned. How many chickens was that?

16 A tailor met a tortoise sitting under a tree. When the tortoise was the tailors age, the tailor was only a quarter of his current age. When the tree was the tortoises age, the tortoise was only a seventh of its current age. If the sum of their ages is now 264, how old is the tortoise?

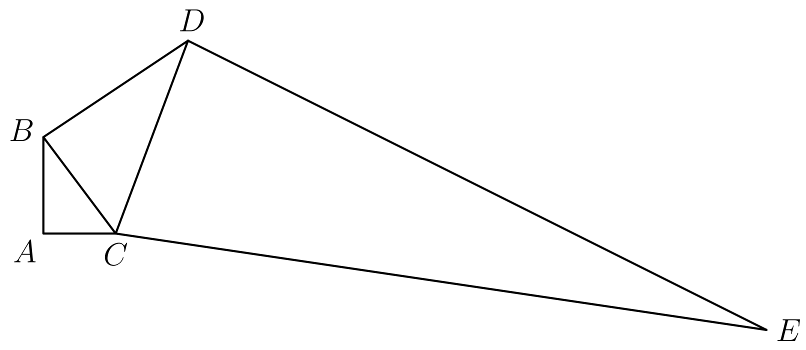
17 Functions f and g are defined so that $f(1) = 4$, $g(1) = 9$, and for each integer $n \geq 1$, $f(n+1) = 2f(n) + 3g(n) + 2n$ and $g(n+1) = 2g(n) + 3f(n) + 5$. Find $f(2005) - g(2005)$.

18 The side lengths of a trapezoid are $\sqrt[4]{3}$, $\sqrt[4]{3}$, $\sqrt[4]{3}$, and $2 \cdot \sqrt[4]{3}$. Its area is the ratio of two relatively prime positive integers, m and n. Find $m + n$.

19 Let x and y be integers satisfying both $x^2 - 16x + 3y = 20$ and $y^2 + 4y - x = -12$. Find $x + y$.

20 The summation $\sum_{k=1}^{360} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}$ is the ratio of two relatively prime positive integers m and n. Find $m + n$.

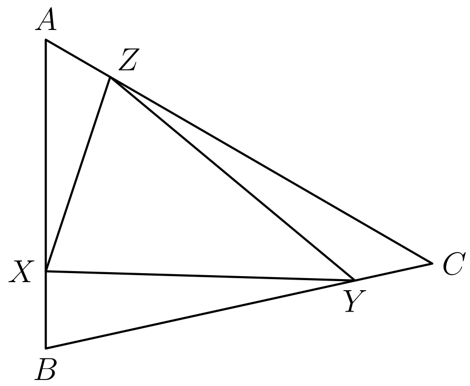
21 In the diagram below $\angle CAB$, $\angle CBD$, and $\angle CDE$ are all right angles with side lengths $AC = 3$, $BC = 5$, $BD = 12$, and $DE = 84$. The distance from point E to the line AB can be expressed as the ratio of two relatively prime positive integers, m and n. Find $m + n$.



22 Let d_k be the greatest odd divisor of k for $k = 1, 2, 3, \ldots$. Find $d_1 + d_2 + d_3 + \ldots + d_{1024}$.

23 Let $a = \sqrt[401]{4} - 1$ and for each $n \geq 2$, let $b_n = \binom{n}{1} + \binom{n}{2}a + \ldots + \binom{n}{n}a^{n-1}$. Find $b_{2006} - b_{2005}$.

24 $\triangle ABC$ has area 240. Points X, Y, Z lie on sides AB, BC, and CA, respectively. Given that $\frac{AX}{BX} = 3$, $\frac{BY}{CY} = 4$, and $\frac{CZ}{AZ} = 5$, find the area of $\triangle XYZ$.



25 Find the number of quadruples (a, b, c, d) of integers which satisfy both

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{2} \qquad \text{and}$$

$$2(a+b+c+d)=ab+cd+(a+b)(c+d)+1.$$