Purple Comet Problems 2006

www.artofproblemsolving.com/community/c4136

by youarebad, btilm305, bluecarneal

– Middle School

---

1 Michael is celebrating his fifteenth birthday today. How many Sundays have there been in his lifetime?

---

2 Evaluate $\frac{\frac{1}{\frac{1}{10}-\frac{1}{12}}}{\frac{1}{\frac{1}{8}-\frac{1}{6}}+\frac{1}{\frac{1}{5}-\frac{1}{6}}}$.

---

3 Find the sum of all the prime numbers less than 100 which are one more than a multiple of six.

---

4 At the beginning of each hour from 1 oclock AM to 12 NOON and from 1 oclock PM to 12 MIDNIGHT a coo-coo clocks coo-coo bird coo-coos the number of times equal to the number of the hour. In addition, the coo-coo clocks coo-coo bird coo-coos a single time at 30 minutes past each hour. How many times does the coo-coo bird coo-coo from $12:42$ PM on Monday until $3:42$ AM on Wednesday?

---

5 The sizes of the freshmen class and the sophomore class are in the ratio $5:4$. The sizes of the sophomore class and the junior class are in the ratio $7:8$. The sizes of the junior class and the senior class are in the ratio $9:7$. If these four classes together have a total of 2158 students, how many of the students are freshmen?

---

6 We draw a radius of a circle. We draw a second radius 23 degrees clockwise from the first radius. We draw a third radius 23 degrees clockwise from the second. This continues until we have drawn 40 radii each 23 degrees clockwise from the one before it. What is the measure in degrees of the smallest angle between any two of these 40 radii?

---

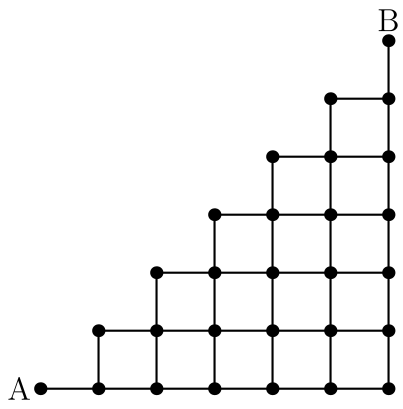
7 At a movie theater tickets for adults cost 4 dollars more than tickets for children. One afternoon the theater sold 100 more child tickets than adult tickets for a total sales amount of 1475 dollars. How much money would the theater have taken in if the same tickets were sold, but the costs of the child tickets and adult tickets were reversed?

---

8 A rogue spaceship escapes. 54 minutes later the police leave in a spaceship in hot pursuit. If the police spaceship travels 12 faster than the rogue spaceship along the same route, how many minutes will it take for the police to catch up with the rogues?

---

9 Moving horizontally and vertically from point to point along the lines in the diagram below, how many routes are there from point A to point B which consist of six horizontal moves and six vertical moves?



---

10 How many rectangles are there in the diagram below such that the sum of the numbers within the
rectangle is a multiple of 7?

43	44	45	46	47	48	49
36	37	38	39	40	41	42
29	30	31	32	33	34	35
22	23	24	25	26	27	28
15	16	17	18	19	20	21
8	9	10	11	12	13	14
1	2	3	4	5	6	7

---

11 Consider the polynomials

$$P(x) = (x + \sqrt{2})(x^2 - 2x + 2)$$
$$Q(x) = (x - \sqrt{2})(x^2 + 2x + 2)$$
$$R(x) = (x^2 + 2)(x^8 + 16).$$

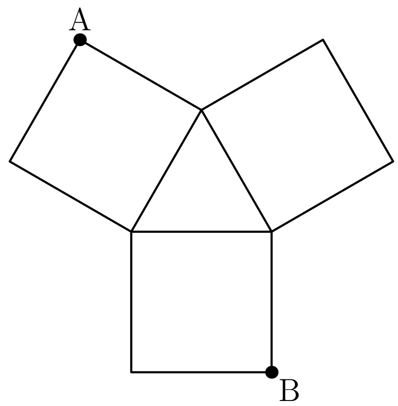
Find the coefficient of x^4 in $P(x) \cdot Q(x) \cdot R(x)$.

---

12 How many positive integers divide the number $10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$?

---

13 An equilateral triangle with side length 6 has a square of side length 6 attached to each of its edges as shown. The distance between the two farthest vertices of this figure (marked A and B in the figure) can be written as $m + \sqrt{n}$ where m and n are positive integers. Find $m + n$.

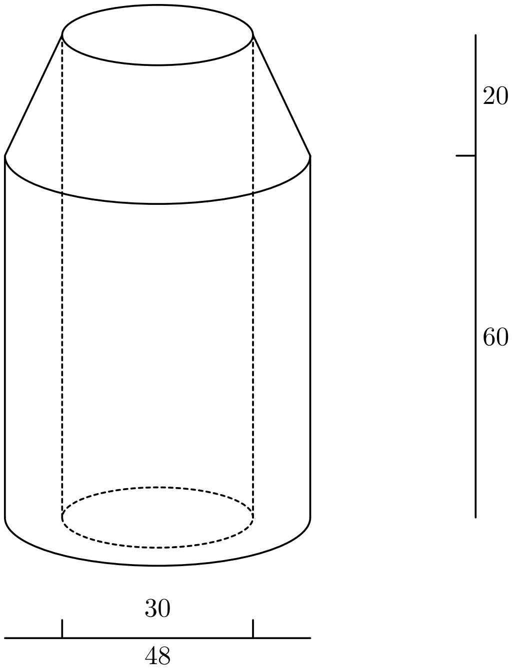


---

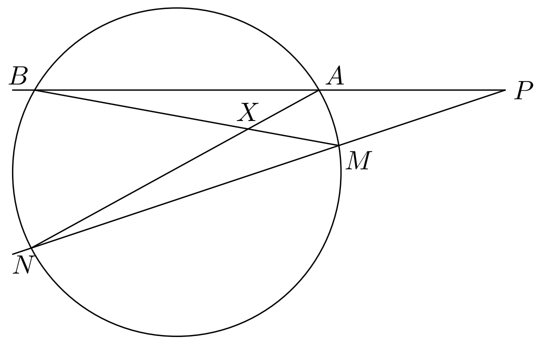
14 The rodent control task force went into the woods one day and caught 200 rabbits and 18 squirrels. The next day they went into the woods and caught 3 fewer rabbits and two more squirrels than the day before. Each day they went into the woods and caught 3 fewer rabbits and two more squirrels than the day before. This continued through the day when they caught more squirrels than rabbits. Up through that day how many rabbits did they catch in all?

---

15 A concrete sewer pipe fitting is shaped like a cylinder with diameter 48 with a cone on top. A cylindrical hole of diameter 30 is bored all the way through the center of the fitting as shown. The cylindrical portion has height 60 while the conical top portion has height 20. Find N such that the volume of the concrete is $N\pi$.



–	High School
1	The sizes of the freshmen class and the sophomore class are in the ratio $5 : 4$. The sizes of the sophomore class and the junior class are in the ratio $7 : 8$. The sizes of the junior class and the senior class are in the ratio $9 : 7$. If these four classes together have a total of 2158 students, how many of the students are freshmen?
2	At a movie theater tickets for adults cost 4 dollars more than tickets for children. One afternoon the theater sold 100 more child tickets than adult tickets for a total sales amount of 1475 dollars. How many dollars would the theater have taken in if the same tickets were sold, but the costs of the child tickets and adult tickets were reversed?
3	Point P lies outside a circle, and two rays are drawn from P that intersect the circle as shown. One ray intersects the circle at points A and B while the other ray intersects the circle at M and N. AN and MB intersect at X. Given that $\angle AXB$ measures 127° and the minor arc AM measures 14°, compute the measure of the angle at P.



4 A rogue spaceship escapes. 54 minutes later the police leave in a spaceship in hot pursuit. If the police spaceship travels 12% faster than the rogue spaceship along the same route, how many minutes will it take for the police to catch up with the rogues?

5 Find the sum of all positive integers less than 2006 which are both multiples of six and one more than a multiple of seven.

6 The positive integers v, w, x, y, and z satisfy the equation

$$v + \cfrac{1}{w + \cfrac{1}{x + \cfrac{1}{y + \cfrac{1}{z}}}} = \frac{222}{155}.$$

Compute $10^4v + 10^3w + 10^2x + 10y + z$.

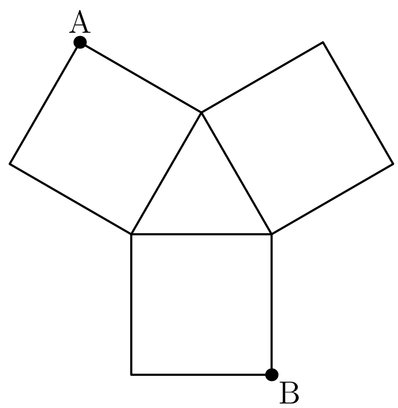
7 Heather and Kyle need to mow a lawn and paint a room. If Heather does both jobs by herself, it will take her a total of nine hours. If Heather mows the lawn and, after she finishes, Kyle paints the room, it will take them a total of eight hours. If Kyle mows the lawn and, after he finishes, Heather paints the room, it will take them a total of seven hours. If Kyle does both jobs by himself, it will take him a total of six hours. It takes Kyle twice as long to paint the room as it does for him to mow the lawn. The number of hours it would take the two of them to complete the two tasks if they worked together to mow the lawn and then worked together to paint the room is a fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

8 Evaluate $\frac{2i}{\frac{1}{30-5i} - \frac{1}{30+5i}}$ where $i = \sqrt{-1}$.

9 How many rectangles are there in the diagram below such that the sum of the numbers within the rectangle is a multiple of 7?

43	44	45	46	47	48	49
36	37	38	39	40	41	42
29	30	31	32	33	34	35
22	23	24	25	26	27	28
15	16	17	18	19	20	21
8	9	10	11	12	13	14
1	2	3	4	5	6	7

10 An equilateral triangle with side length 6 has a square of side length 6 attached to each of its edges as shown. The distance between the two farthest vertices of this figure (marked A and B in the figure) can be written as $m+\sqrt{n}$ where m and n are positive integers. Find $m+n$.



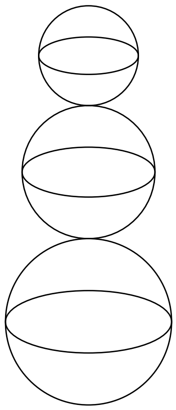
11 Let k be the product of every third positive integer from 2 to 2006, that is $k=2\cdot5\cdot8\cdot11\cdots2006$. Find the number of zeros there are at the right end of the decimal representation for k.

12 We draw a triangle inside of a circle with one vertex at the center of the circle and the other two vertices on the circumference of the circle. The angle at the center of the circle measures 75 degrees. We draw a second triangle, congruent to the first, also with one vertex at the center of the circle and the other vertices on the circumference of the circle rotated 75 degrees clockwise from the first triangle so that it shares a side with the first triangle. We draw a third, fourth, and fifth such triangle each rotated 75 degrees clockwise from the previous triangle. The base of the fifth triangle will intersect the base of the first triangle. What is the degree measure of the obtuse angle formed by the intersection?

13 12 students need to form five study groups. They will form three study groups with 2 students each and two study groups with 3 students each. In how many ways can these groups be formed?

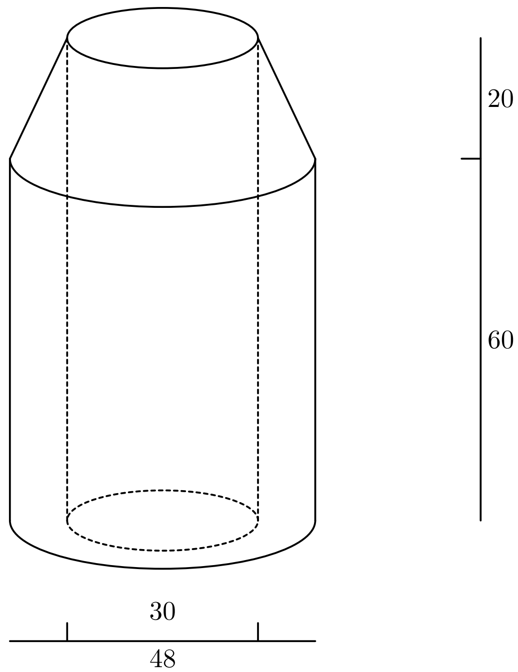
14 Consider all ordered pairs (m, n) of positive integers satisfying $59m - 68n = mn$. Find the sum of all the possible values of n in these ordered pairs.

15 A snowman is built on a level plane by placing a ball radius 6 on top of a ball radius 8 on top of a ball radius 10 as shown. If the average height above the plane of a point in the snowman is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.



16 $f(x)$ and $g(x)$ are linear functions such that for all x, $f(g(x)) = g(f(x)) = x$. If $f(0) = 4$ and $g(5) = 17$, compute $f(2006)$.

17 A concrete sewer pipe fitting is shaped like a cylinder with diameter 48 with a cone on top. A cylindrical hole of diameter 30 is bored all the way through the center of the fitting as shown. The cylindrical portion has height 60 while the conical top portion has height 20. Find N such that the volume of the concrete is $N\pi$.



18 In how many ways can 100 be written as the sum of three positive integers x, y, and z satisfying $x < y < z$?

19 There is a very popular race course where runners frequently go for a daily run. Assume that all runners randomly select a start time, a starting position on the course, and a direction to run. Also assume that all runners make exactly one complete circuit of the race course, all runners run at the same speed, and all runners complete the circuit in one hour. Suppose that one afternoon you go for a run on this race course, and you count 300 runners which you pass going in the opposite direction, although some of those runners you count twice since you pass them twice. What is the expected value of the number of different runners that you pass not counting duplicates?

20 Find the sum of all the positive integers which have at most three not necessarily distinct prime factors where the primes come from the set $\{2, 3, 5, 7\}$.

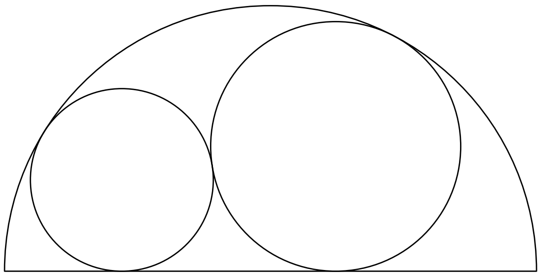
21 In triangle ABC, $AB = 52$, $BC = 56$, $CA = 60$. Let D be the foot of the altitude from A and E be the intersection of the internal angle bisector of $\angle BAC$ with BC. Find DE.

22 Let $F_0 = 0, F_1 = 1$, and for $n \geq 1, F_{n+1} = F_n + F_{n-1}$. Define $a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot F_n$. Then there are rational numbers A and B such that $\frac{a_{30}+a_{29}}{a_{26}+a_{25}} = A + B\sqrt{5}$. Find $A + B$.

23 We have two positive integers both less than 1000. The arithmetic mean and the geometric mean of these numbers are consecutive odd integers. Find the maximum possible value of the difference of the two integers.

---

24 A semicircle with diameter length 16 contains a circle radius 3 tangent both to the inside of the semicircle and its diameter as shown. A second larger circle is tangent to the inside of the semicircle, the outside of the circle, and the diameter of the semicircle. The diameter of the second circle can be written as $\frac{n+k\sqrt{2}}{m}$ where m, n, and k are positive integers and m and n have no factors in common. Find $m+n+k$.



---

25 Let x and y be two real numbers such that $2\sin x \sin y + 3\cos y + 6\cos x \sin y = 7$. Find $\tan^2 x + 2\tan^2 y$.
