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Purple Comet Problems 2007

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–	Middle School
1	Last Sunday at noon the date on the calendar was 15 (April 15, 2007). What will be the date on the calendar one million minutes after that time?
2	How many numbers n have the property that both $\frac{n}{2}$ and $2n$ are four digits whole numbers?
3	Square $ABCD$ has side length 36. Point E is on side AB a distance 12 from B, point F is the midpoint of side BC, and point G is on side CD a distance 12 from C. Find the area of the region that lies inside triangle EFG and outside triangle AFD.
4	Terry drove along a scenic road using 9 gallons of gasoline. Then Terry went onto the freeway and used 17 gallons of gasoline. Assuming that Terry gets 6.5 miles per gallon better gas mileage on the freeway than on the scenic road, and Terrys average gas mileage for the entire trip was 30 miles per gallon, find the number of miles Terry drove.
5	The repeating decimal $0.328181818181...$ can equivalently be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.
6	The product of two positive numbers is equal to 50 times their sum and 75 times their difference. Find their sum.
7	There is an interval $[a,b]$ that is the solution to the inequality $$\|3x-80\| \le \|2x-105\|$$ Find $a+b$.
8	Penelope plays a game where she adds 25 points to her score each time she wins a game and deducts 13 points from her score each time she loses a game. Starting with a score of zero, Penelope plays m games and has a total score of 2007 points. What is the smallest possible value for m?
9	Purple College keeps a careful count of its students as they progress each year from the freshman class to the sophomore class to the junior class and, finally, to the senior class. Each year at the college one third of the freshman class drops out of school, 40 students in the sophomore class drop out of school, and one tenth of the junior class drops out of school. Given that the college only admits new freshman students, and that it wants to begin each school

year with 3400 students enrolled, how many students does it need to admit into the freshman class each year?

10 Tom can run to Beth's house in 63 minutes. Beth can run to Tom's house in 84 minutes. At noon Tom starts running from his house toward Beth's house while at the same time Beth starts running from her house toward Tom's house. When they meet, they both run at Beth's speed back to Beth's house. At how many minutes after noon will they arrive at Beth's house?

11 The alphabet in its natural order ABCDEFGHIJKLMNOPQRSTUVWXYZ is T_0. We apply a permutation to T_0 to get T_1 which is JQOWIPANTZRCVMYEGSHUFDKBLX. If we apply the same permutation to T_1, we get T_2 which is ZGYKTEJMUXSODVLIAHNFPWRQCB. We continually apply this permutation to each T_m to get T_{m+1}. Find the smallest positive integer n so that $T_n = T_0$.

12 If you alphabetize all of the distinguishable rearrangements of the letters in the word **PURPLE**, find the number n such that the word **PURPLE** is the nth item in the list.

13 Evaluate the sum

$$1^2 + 2^2 - 3^2 - 4^2 + 5^2 + 6^2 - 7^2 - 8^2 + \cdots - 1000^2 + 1001^2$$

15 We have some identical paper squares which are black on one side of the sheet and white on the other side. We can join nine squares together to make a 3 by 3 sheet of squares by placing each of the nine squares either white side up or black side up. Two of these 3 by 3 sheets are distinguishable if neither can be made to look like the other by rotating the sheet or by turning it over. How many distinguishable 3 by 3 squares can we form?

– High School

1 The sum of nine consecutive odd numbers is 2007. Find the greatest of these nine numbers.

2 A positive number $\dfrac{m}{n}$ has the property that it is equal to the ratio of 7 plus the numbers reciprocal and 65 minus the numbers reciprocal. Given that m and n are relatively prime positive integers, find $2m + n$.

3 A bowl contained 10% blue candies and 25% red candies. A bag containing three quarters red candies and one quarter blue candies was added to the bowl. Now the bowl is 16% blue candies. What percentage of the candies in the bowl are now red?

4 To the nearest degree, find the measure of the largest angle in a triangle with side lengths 3, 5, and 7.

5 $F(0) = 3$ and $F(n) = F(n-1) + 4$ when n is positive. Find $F(F(F(5)))$.

6 Find the sum of all the positive integers that are divisors of either 96 or 180.

7 Allowing x to be a real number, what is the largest value that can be obtained by the function $25\sin(4x) - 60\cos(4x)$?

8 You know that the Jones family has five children, and the Smith family has three children. Of the eight children you know that there are five girls and three boys. Let $\frac{m}{n}$ be the probability that at least one of the families has only girls for children. Given that m and n are relatively prime positive integers, find $m + n$.

9 The four sets A, B, C, and D each have 400 elements. The intersection of any two of the sets has 115 elements. The intersection of any three of the sets has 53 elements. The intersection of all four sets has 28 elements. How many elements are there in the union of the four sets?

10 For a particular value of the angle θ we can take the product of the two complex numbers $(8+i)\sin\theta + (7+4i)\cos\theta$ and $(1+8i)\sin\theta + (4+7i)\cos\theta$ to get a complex number in the form $a + bi$ where a and b are real numbers. Find the largest value for $a + b$.

11 A dart board looks like three concentric circles with radii of 4, 6, and 8. Three darts are thrown at the board so that they stick at three random locations on then board. The probability that one dart sticks in each of the three regions of the dart board is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

12 Find the maximum possible value of $8 \cdot 27^{\log_6 x} + 27 \cdot 8^{\log_6 x} - x^3$ as x varies over the positive real numbers.

13 Find the circumradius of the triangle with side lengths 104, 112, and 120.

14 A rectangular storage bin measures 10 feet by 12 feet, is 3 feet tall, and sits on a flat plane. A pile of dirt is pushed up against the outside of the storage bin so that it slants down from the top of the storage bin to points on the ground 4 feet away from the base of the storage bin as shown. The number of cubic feet of dirt needed to form the pile can be written as $m + n\pi$ where m and n are positive integers. Find $m + n$.

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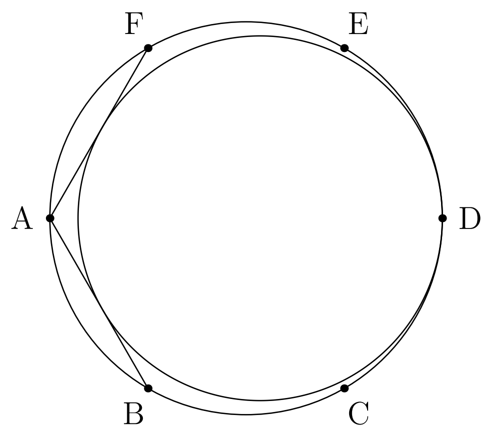
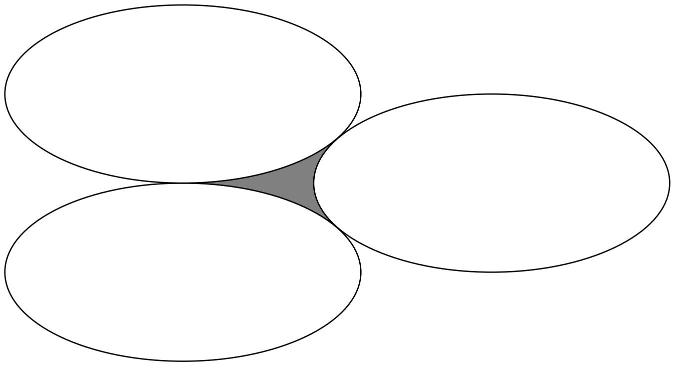
17 A circle with diameter 20 has points $A, B, C, D, E,$ and F equally spaced along its circumference. A second circle is tangent to the lines AB and AF and internally tangent to the circle. If the second circle has diameter $\sqrt{m} + n$ for integers m and n, find $m + n$.



18 Let S be the graph of $y = x^3$, and T be the graph of $y = \sqrt[3]{y}$. Let S^* be S rotated around the origin 15 degrees clockwise, and T^* be T rotated around the origin 45 degrees counterclockwise. S^* and T^* will intersect at a point in the first quadrant a distance $M + \sqrt{N}$ from the origin where M and N are positive integers. Find $M + N$.

19 Six chairs sit in a row. Six people randomly seat themselves in the chairs. Each person randomly chooses either to set their feet on the floor, to cross their legs to the right, or to cross their legs to the left. There is only a problem if two people sitting next to each other have the person on the right crossing their legs to the left and the person on the left crossing their legs to the right. The probability that this will **not** happen is given by $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

20 Three congruent ellipses are mutually tangent. Their major axes are parallel. Two of the ellipses are tangent at the end points of their minor axes as shown. The distance between the centers of these two ellipses is 4. The distances from those two centers to the center of the third ellipse are both 14. There are positive integers m and n so that the area between these three ellipses is $\sqrt{n} - m\pi$. Find $m + n$.



21 What is the greatest positive integer m such that $n^2(1+n^2-n^4) \equiv 1 \pmod{2^m}$ for all odd integers n?

22 Let $a = 3^{1/223} + 1$ and for all $n \geq 3$ let

$$f(n) = \binom{n}{0}a^{n-1} - \binom{n}{1}a^{n-2} + \binom{n}{2}a^{n-3} - ... + (-1)^{n-1}\binom{n}{n-1}a^0.$$

Find $f(2007) + f(2008)$.

23 Two circles with radius 2 and radius 4 have a common center at P. Points $A, B,$ and C on the larger circle are the vertices of an equilateral triangle. Point D is the intersection of the smaller circle and the line segment PB. Find the square of the area of triangle ADC.

24 Starting with a sequence of $n 1's$, you can insert plus signs to get various sums. For example, when $n = 10$, you can get the sum $1+1+1+11+11+111 = 136$, and the sum $1+1+11+111+111 = 235$. Find the number of values of n so that the sum of 1111 is possible.

25 Let x be a positive integer less than 200, and let y be obtained by writing the base 10 digits of x in reverse order. Given that x and y satisfy $11x^2 + 363y = 7xy + 6571$, find x.