Purple Comet Problems 2009

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– High School

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1 In five years, Tom will be twice as old as Cindy. Thirteen years ago, Tom was three times as old as Cindy. How many years ago was Tom four times as old as Cindy?

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2 Find the least positive integer n such that for every prime number $p, p^2 + n$ is never prime.

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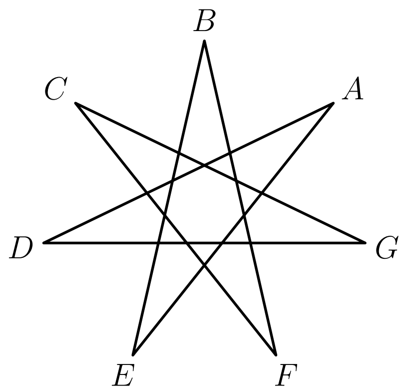
3 In the diagram $ABCDEFG$ is a regular heptagon (a 7 sided polygon). Shown is the star $AEBFCGD$. The degree measure of the obtuse angle formed by AE and CG is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.



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4 There are three bags of marbles. Bag two has twice as many marbles as bag one. Bag three has three times as many marbles as bag one. Half the marbles in bag one, one third the marbles in bag two, and one fourth the marbles in bag three are green. If all three bags of marbles are dumped into a single pile, $\frac{m}{n}$ of the marbles in the pile would be green where m and n are relatively prime positive integers.
Find $m + n$.

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5 Find n so that $(4^{n+7})^3 = (2^{n+23})^4$.

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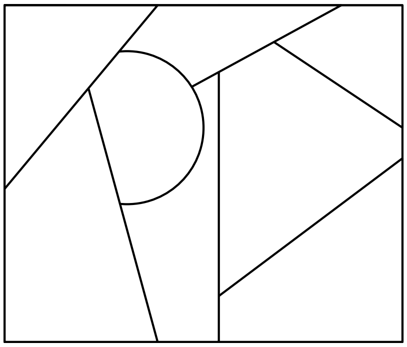
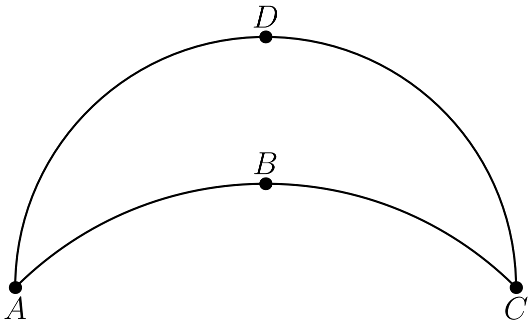
6 Wiles county contains eight townships as shown on the map. If there are four colors available, in how many ways can the the map be colored so that each township is colored with one color and no two townships that share a border are colored with the same color?



7 The figure $ABCD$ is bounded by a semicircle CDA and a quarter circle ABC. Given that the distance from A to C is 18, find the area of the figure.



8 Find the least positive integer that has exactly 20 positive integer divisors.

9 Bill bought 13 notebooks, 26 pens, and 19 markers for 25 dollars. Paula bought 27 notebooks, 18 pens, and 31 markers for 31 dollars. How many dollars would it cost Greg to buy 24 notebooks, 120 pens, and 52 markers?

10 Towers grow at points along a line. All towers start with height 0 and grow at the rate of one meter per second. As soon as any two adjacent towers are each at least 1 meter tall, a new tower begins to grow at a point along the line exactly half way between those two adjacent towers. Before time 0 there are no towers, but at time 0 the first two towers begin to grow at two points along the line. Find the total of all the heights of all towers at time 10 seconds.

11 The four points $A(-1,2), B(3,-4), C(5,-6)$, and $D(-2,8)$ lie in the coordinate plane. Compute the minimum possible value of $PA+PB+PC+PD$ over all points P .

12 What is the least possible sum of two positive integers a and b where $a \cdot b = 10!$?

13 Greta is completing an art project. She has twelve sheets of paper: four red, four white, and four blue. She also has twelve paper stars: four red, four white, and four blue. She randomly places one star on each sheet of paper. The probability that no star will be placed on a sheet of paper that is the same color as the star is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $n - 100m$.

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14 Let $ABCD$ be a trapezoid with AB parallel to CD, AB has length 1, and CD has length 41. Let points X and Y lie on sides AD and BC, respectively, such that XY is parallel to AB and CD, and XY has length 31. Let m and n be two relatively prime positive integers such that the ratio of the area of $ABYX$ to the area of $CDXY$ is $\frac{m}{n}$. Find $m + 2n$.

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15 What is the remainder when 7^{8^9} is divided by $1000?$

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16 Let the complex number $z = \cos\frac{1}{1000} + i\sin\frac{1}{1000}$. Find the smallest positive integer n so that z^n has an imaginary part which exceeds $\frac{1}{2}$.

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17 How many ordered triples (a, b, c) of odd positive integers satisfy $a + b + c = 25?$

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18 On triangle ABC let D be the point on AB so that CD is an altitude of the triangle, and E be the point on BC so that AE bisects angle BAC. Let G be the intersection of AE and CD, and let point F be the intersection of side AC and the ray BG. If AB has length 28, AC has length 14, and CD has length 10, then the length of CF can be written as $\frac{k-m\sqrt{p}}{n}$ where $k, m, n,$ and p are positive integers, k and n are relatively prime, and p is not divisible by the square of any prime. Find $k - m + n + p$.

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19 If a and b are complex numbers such that $a^2 + b^2 = 5$ and $a^3 + b^3 = 7$, then their sum, $a + b$, is real. The greatest possible value for the sum $a + b$ is $\frac{m+\sqrt{n}}{2}$ where m and n are integers. Find n.

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20 Five men and seven women stand in a line in random order. Let m and n be relatively prime positive integers so that $\frac{m}{n}$ is the probability that each man stands next to at least one woman. Find $m + n$.

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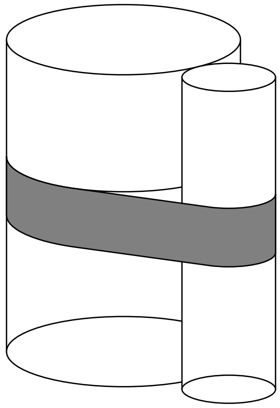
21 A cylinder radius 12 and a cylinder radius 36 are held tangent to each other with a tight band. The length of the band is $m\sqrt{k}+n\pi$ where m, k, and n are positive integers, and k is not divisible by the square of any prime. Find $m + k + n$.



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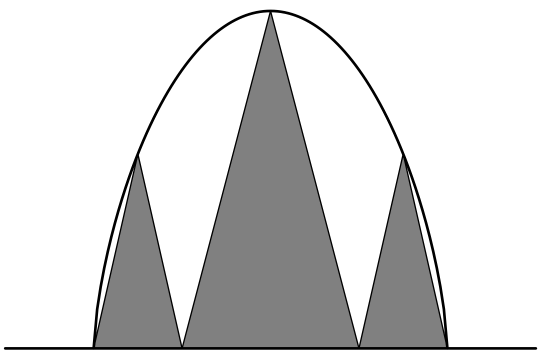
22 The diagram shows a parabola, a line perpendicular to the parabola's axis of symmetry, and three similar isosceles triangles each with a base on the line and vertex on the parabola. The two smaller triangles are congruent and each have one base vertex on the parabola and one base vertex shared with the larger triangle. The ratio of the height of the larger triangle to the height of the smaller triangles is $\frac{a+\sqrt{b}}{c}$ where a, b, and c are positive integers, and a and c are relatively prime. Find $a+b+c$.



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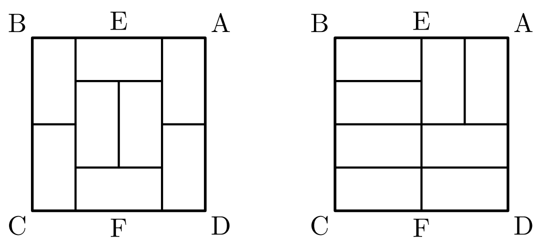
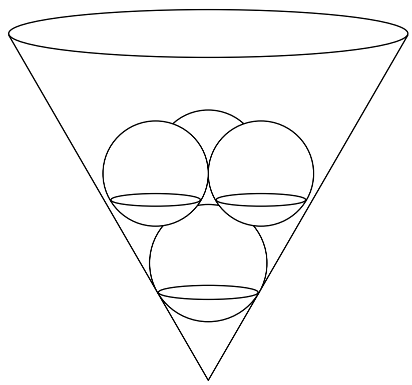
23 Square $ABCD$ has side length 4. Points E and F are the midpoints of sides AB and CD, respectively. Eight 1 by 2 rectangles are placed inside the square so that no two of the eight rectangles overlap (see diagram). If the arrangement of eight rectangles is chosen randomly, then there are relatively prime positive integers m and n so that $\frac{m}{n}$ is the probability that none of the rectangles crosses the line segment EF (as in the arrangement on the right). Find $m+n$.



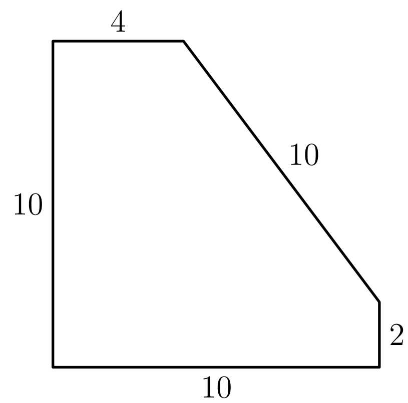
24 A right circular cone pointing downward forms an angle of 60° at its vertex. Sphere S with radius 1 is set into the cone so that it is tangent to the side of the cone. Three congruent spheres are placed in the cone on top of S so that they are all tangent to each other, to sphere S, and to the side of the cone. The radius of these congruent spheres can be written as $\frac{a+\sqrt{b}}{c}$ where a, b, and c are positive integers such that a and c are relatively prime. Find $a+b+c$.



25 The polynomial $P(x) = a_0 + a_1x + a_2x^2 + ... + a_8x^8 + 2009x^9$ has the property that $P(\frac{1}{k}) = \frac{1}{k}$ for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9$. There are relatively prime positive integers m and n such that $P(\frac{1}{10}) = \frac{m}{n}$. Find $n - 10m$.

– Middle School

1 The pentagon below has three right angles. Find its area.



2 Let $p_1 = 2, p_2 = 3, p_3 = 5...$ be the sequence of prime numbers. Find the least positive even integer n so that $p_1 + p_2 + p_3 + ... + p_n$ is not prime.

3 The *Purple Comet! Math Meet* runs from April 27 through May 3, so the sum of the calendar dates for these seven days is $27 + 28 + 29 + 30 + 1 + 2 + 3 = 120$. What is the largest sum of the calendar dates for seven consecutive Fridays occurring at any time in any year?

4 John, Paul, George, and Ringo baked a circular pie. Each cut a piece that was a sector of the circle. John took one-third of the whole pie. Paul took one-fourth of the whole pie. George took one-fifth of the whole pie. Ringo took one-sixth of the whole pie. At the end the pie had one sector remaining. Find the measure in degrees of the angle formed by this remaining sector.

5 A train car held 6000 pounds of mud which was 88 percent water. Then the train car sat in the sun, and some of the water evaporated so that now the mud is only 82 percent water. How many pounds does the mud weigh now?

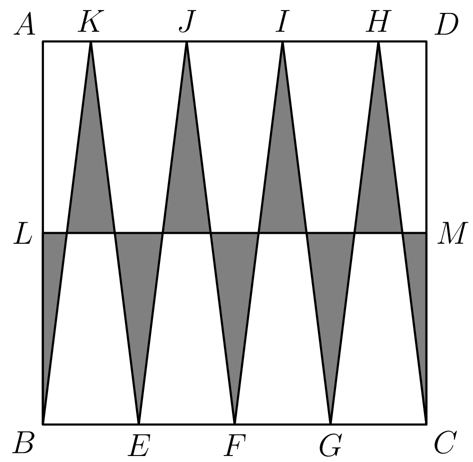
6 Find n such that $20^{2009} = 10^{2000} \cdot 40^9 \cdot 2^n$.

7 How many distinct four letter arrangements can be formed by rearranging the letters found in the word **FLUFFY**? For example, FLYF and ULFY are two possible arrangements.

8 Find the number of non-congruent scalene triangles whose sides all have integral length, and the longest side has length 11.

9 One plant is now 44 centimeters tall and will grow at a rate of 3 centimeters every 2 years. A second plant is now 80 centimeters tall and will grow at a rate of 5 centimeters every 6 years. In how many years will the plants be the same height?

10 The diagram shows a 20 by 20 square $ABCD$. The points E, F, and G are equally spaced on side BC. The points H, I, J, and K on side DA are placed so that the triangles BKE, EJF, FIG, and GHC are isosceles. Points L and M are midpoints of the sides AB and CD, respectively. Find the total area of the shaded regions.

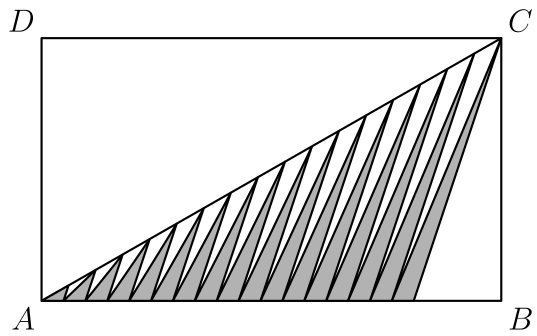


11 Aisha went shopping. At the first store she spent 40 percent of her money plus four dollars. At the second store she spent 50 percent of her remaining money plus 5 dollars. At the third store she spent 60 percent of her remaining money plus six dollars. When Aisha was done shopping at the three stores, she had two dollars left. How many dollars did she have with her when she started shopping?

12 In isosceles triangle ABC sides AB and BC have length 125 while side AC has length 150. Point D is the midpoint of side AC. E is on side BC so that BC and DE are perpendicular. Similarly, F is on side AB so that AB and DF are perpendicular. Find the area of triangle DEF.

13 How many subsets of the set $\{1, 2, 3, \ldots, 12\}$ contain exactly one or two prime numbers?

14 Rectangle $ABCD$ measures 70 by 40. Eighteen points (including A and C) are marked on the diagonal AC dividing the diagonal into 17 congruent pieces. Twenty-two points (including A and B) are marked on the side AB dividing the side into 21 congruent pieces. Seventeen non-overlapping triangles are constructed as shown. Each triangle has two vertices that are two of these adjacent marked points on the side of the rectangle, and one vertex that is one of the marked points along the diagonal of the rectangle. Only the left 17 of the 21 congruent pieces along the side of the rectangle are used as bases of these triangles. Find the sum of the areas of these 17 triangles.



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15 We have twenty-seven 1 by 1 cubes. Each face of every cube is marked with a natural number so that two opposite faces (top and bottom, front and back, left and right) are always marked with an even number and an odd number where the even number is twice that of the odd number. The twenty-seven cubes are put together to form one 3 by 3 cube as shown. When two cubes are placed face-to-face, adjoining faces are always marked with an odd number and an even number where the even number is one greater than the odd number. Find the sum of all of the numbers on all of the faces of all the 1 by 1 cubes.