**Purple Comet Problems 2010**

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by KingSmasher3, djmathman, Mrdavid445, El_Ectric, 1-1 is 3

| – | Middle School |
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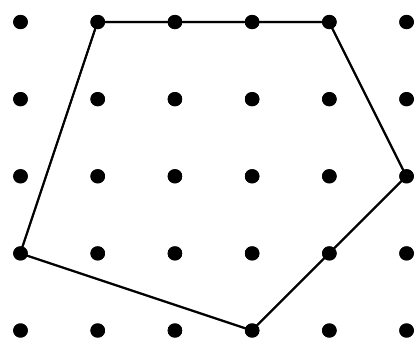
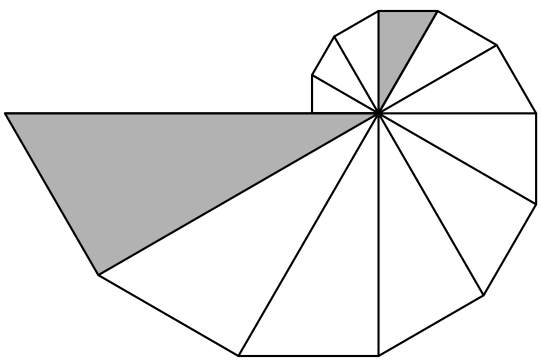
| 1 | If $125 + n + 135 + 2n + 145 = 900$, find $n$. |
| 2 | Three boxes each contain four bags. Each bag contains five marbles. How many marbles are there altogether in the three boxes? |
| 3 | The sum $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m + n$. |
| 4 | The grid below contains five rows with six points in each row. Points that are adjacent either horizontally or vertically are a distance one apart. Find the area of the pentagon shown.  |
| 5 | Find the least positive integer $k$ so that $k + 25973$ is a palindrome (a number which reads the same forward and backwards). |
| 6 | Find the sum of the prime factors of $777$. |
| 7 | $x$ and $y$ are positive real numbers where $x$ is $p$ percent of $y$, and $y$ is $4p$ percent of $x$. What is $p$? |
| 8 | There are exactly two four-digit numbers that are multiples of three where their first digit is double their second digit, their third digit is three more than their fourth digit, and their second digit is $2$ less than their fourth digit. Find the difference of these two numbers. |
| 9 | What percent of the numbers $1, 2, 3, ...1000$ are divisible by exactly one of the numbers $4$ and $5?$ |

**10** A baker uses $6\frac{2}{3}$ cups of flour when she prepares $\frac{5}{3}$ recipes of rolls. She will use $9\frac{3}{4}$ cups of flour when she prepares $\frac{m}{n}$ recipes of rolls where m and n are relatively prime positive integers. Find $m+n$.

**11** There are two rows of seats with three side-by-side seats in each row. Two little boys, two little girls, and two adults sit in the six seats so that neither little boy sits to the side of either little girl. In how many different ways can these six people be seated?

**12** The diagram below shows twelve $30-60-90$ triangles placed in a circle so that the hypotenuse of each triangle coincides with the longer leg of the next triangle. The fourth and last triangle in this diagram are shaded. The ratio of the perimeters of these two triangles can be written as $\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m+n$.



**13** Find the number of sets $A$ that satisfy the three conditions: $\star$ $A$ is a set of two positive integers $\star$ each of the numbers in $A$ is at least $22$ percent the size of the other number $\star$ $A$ contains the number $30$.

**14** Let $ABCD$ be a trapezoid where $AB$ is parallel to $CD$. Let $P$ be the intersection of diagonal $AC$ and diagonal $BD$. If the area of triangle $PAB$ is $16$, and the area of triangle $PCD$ is $25$, find the area of the trapezoid.

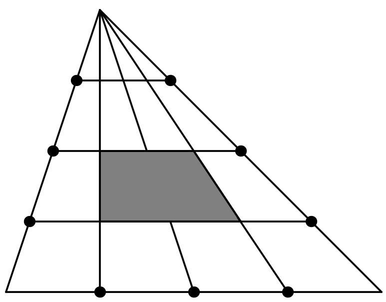
**15** In the number arrangement

$$\begin{array}{lllll}
1 \\
2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 \\
\vdots
\end{array}$$

what is the number that will appear directly below the number $2010$?

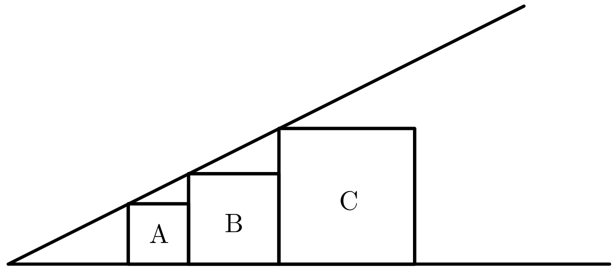
**16** Half the volume of a 12 foot high cone-shaped pile is grade A ore while the other half is grade B ore. The pile is worth $62. One-third of the volume of a similarly shaped 18 foot pile is grade A ore while the other two-thirds is grade B ore. The second pile is worth $162. Two-thirds of the volume of a similarly shaped 24 foot pile is grade A ore while the other one-third is grade B ore. What is the value in dollars ($) of the 24 foot pile?

**17** The diagram below shows a triangle divided into sections by three horizontal lines which divide the altitude of the triangle into four equal parts, and three lines connecting the top vertex with points that divide the opposite side into four equal parts. If the shaded region has area $100$, find the area of the entire triangle.



**18** How many three-digit positive integers contain both even and odd digits?

**19** Square $A$ is adjacent to square $B$ which is adjacent to square $C$. The three squares all have their bottom sides along a common horizontal line. The upper left vertices of the three squares are collinear. If square $A$ has area $24$, and square $B$ has area $36$, find the area of square $C$.



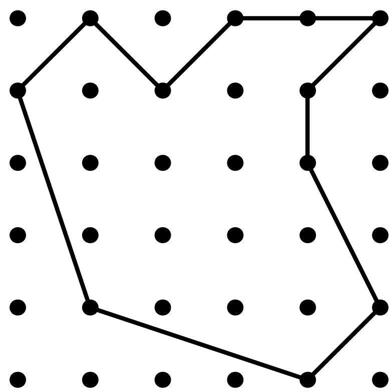

**20** Suppose that $f$ is a function such that $3f(x) - 5xf\left(\frac{1}{x}\right) = x - 7$ for all non-zero real numbers $x$. Find $f(2010)$.

**–** High School

**1** Let $x$ satisfy $(6x+7)+(8x+9)=(10+11x)+(12+13x)$. There are relatively prime positive integers so that $x=-\frac{m}{n}$. Find $m+n$.

**2** The prime factorization of $12 = 2 \cdot 2 \cdot 3$ has three prime factors. Find the number of prime factors in the factorization of $12! = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

**3** The grid below contains six rows with six points in each row. Points that are adjacent either horizontally or vertically are a distance two apart. Find the area of the irregularly shaped ten sided figure shown.



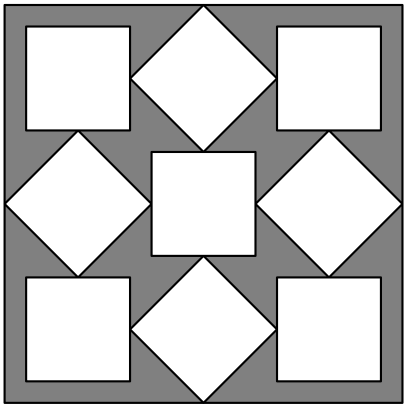
**4** Sally's salary in 2006 was $\$37,500$. For 2007 she got a salary increase of $x$ percent. For 2008 she got another salary increase of $x$ percent. For 2009 she got a salary decrease of $2x$ percent. Her 2009 salary is $\$34,825$. Suppose instead, Sally had gotten a $2x$ percent salary decrease for 2007, an $x$ percent salary increase for 2008, and an $x$ percent salary increase for 2009. What would her 2009 salary be then?

**5** If $a$ and $b$ are positive integers such that $a \cdot b = 2400$, find the least possible value of $a+b$.

**6** Evaluate the sum $1+2-3+4+5-6+7+8-9\cdots+208+209-210$.

**7** Find the sum of the digits in the decimal representation of the number $5^{2010} \cdot 16^{502}$.

**8** The diagram below shows some small squares each with area $3$ enclosed inside a larger square. Squares that touch each other do so with the corner of one square coinciding with the midpoint of a side of the other square. Find integer $n$ such that the area of the shaded region inside the larger square but outside the smaller squares is $\sqrt{n}$.



**9** Find positive integer $n$ so that $\frac{80-6\sqrt{n}}{n}$ is the reciprocal of $\frac{80+6\sqrt{n}}{n}$.

**10** The set $S$ contains nine numbers. The mean of the numbers in $S$ is $202$. The mean of the five smallest of the numbers in $S$ is $100$. The mean of the five largest numbers in $S$ is $300$. What is the median of the numbers in $S$?

**11** A jar contains one white marble, two blue marbles, three red marbles, and four green marbles. If you select two of these marbles without replacement, the probability that both marbles will be the same color is $\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

**12** A good approximation of $\pi$ is $3.14$. Find the least positive integer $d$ such that if the area of a circle with diameter $d$ is calculated using the approximation $3.14$, the error will exceed $1$.

**13** Let $S$ be the set of all $10-$term arithmetic progressions that include the numbers $4$ and $10$. For example, $(-2,1,4,7,10,13,16,19,22,25)$ and $(10,8\frac{1}{2},7,5\frac{1}{2},4,2\frac{1}{2},1,\frac{1}{2},-2,-3\frac{1}{2})$ are both members of $S$. Find the sum of all values of $a_{10}$ for each $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10}) \in S$, that is, $\sum_{a_1,a_2,a_3,\ldots,a_{10}\in S} a_{10}$.

**14** There are positive integers $b$ and $c$ such that the polynomial $2x^2+bx+c$ has two real roots which differ by $30$. Find the least possible value of $b+c$.

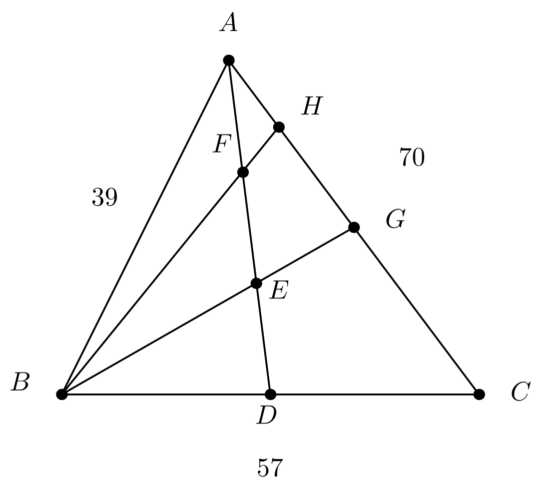
**15** Find the smallest possible sum $a+b+c+d+e$ where $a,b,c,d,$ and $e$ are positive integers satisfying the conditions

$\star$ each of the pairs of integers $(a,b),(b,c),(c,d),$ and $(d,e)$ are **not** relatively prime

$\star$ all other pairs of the five integers **are** relatively prime.

**16** The triangle $ABC$ has sides lengths $AB=39$, $BC=57$, and $CA=70$ as shown. Median $\overline{AD}$ is divided into three congruent segments by points $E$ and $F$. Lines $BE$ and $BF$ intersect side

$\overline{AC}$ at points $G$ and $H$, respectively. Find the distance from $G$ to $H$.




**17** Alan, Barb, Cory, and Doug are on the golf team, Doug, Emma, Fran, and Greg are on the swim team, and Greg, Hope, Inga, and Alan are on the tennis team. These nine people sit in a circle in random order. The probability that no two people from the same team sit next to each other is $\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

**18** When $4\cos\theta - 3\sin\theta = \frac{13}{3}$, it follows that $7\cos 2\theta - 24\sin 2\theta = \frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

**19** The centers of the three circles A, B, and C are collinear with the center of circle B lying between the centers of circles A and C. Circles A and C are both externally tangent to circle B, and the three circles share a common tangent line. Given that circle A has radius $12$ and circle B has radius $42$, find the radius of circle C.

**20** How many of the rearrangements of the digits $123456$ have the property that for each digit, no more than two digits smaller than that digit appear to the right of that digit? For example, the rearrangement $315426$ has this property because digits $1$ and $2$ are the only digits smaller than $3$ which follow $3$, digits $2$ and $4$ are the only digits smaller than $5$ which follow $5$, and digit $2$ is the only digit smaller than $4$ which follows $4$.

**21** Let $a$ be the sum of the numbers:

$99 \times 0.9$

$999 \times 0.9$

$9999 \times 0.9$

$\vdots$

$999\cdots 9 \times 0.9$

where the final number in the list is $0.9$ times a number written as a string of $101$ digits all equal to $9$.
Find the sum of the digits in the number $a$.

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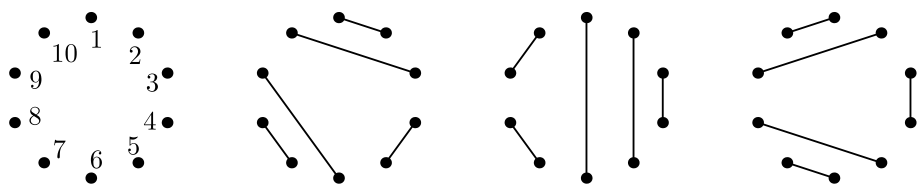
**22** Ten distinct points are placed on a circle. All ten of the points are paired so that the line segments connecting the pairs do not intersect. In how many different ways can this pairing be done?

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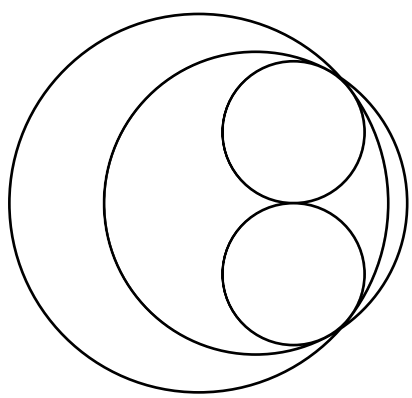
**23** A disk with radius $10$ and a disk with radius $8$ are drawn so that the distance between their centers is $3$. Two congruent small circles lie in the intersection of the two disks so that they are tangent to each other and to each of the larger circles as shown. The radii of the smaller circles are both $\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m+n$.



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**24** Find the number of ordered pairs of integers $(m, n)$ that satisfy $20m - 10n = mn$.

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**25** Let $x_1$, $x_2$, and $x_3$ be the roots of the polynomial $x^3 + 3x + 1$. There are relatively prime positive integers $m$ and $n$ such that $\frac{m}{n} = \frac{x_1^2}{(5x_2+1)(5x_3+1)} + \frac{x_2^2}{(5x_1+1)(5x_3+1)} + \frac{x_3^2}{(5x_1+1)(5x_2+1)}$. Find $m+n$.

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**26** In the coordinate plane a parabola passes through the points $(7, 6)$, $(7, 12)$, $(18, 19)$, and $(18, 48)$. The axis of symmetry of the parabola is a line with slope $\frac{r}{s}$ where r and s are relatively prime positive integers. Find $r + s$.

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**27** Let $a$ and $b$ be real numbers satisfying $2(\sin a + \cos a) \sin b = 3 - \cos b$. Find $3 \tan^2 a + 4 \tan^2 b$.

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**28** There are relatively prime positive integers $p$ and $q$ such that $\dfrac{p}{q} = \displaystyle\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$. Find $p+q$.

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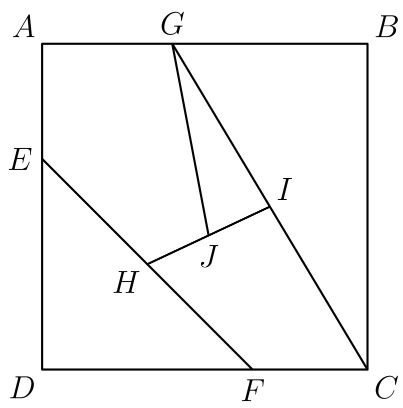
**29** Square $ABCD$ is shown in the diagram below. Points $E$, $F$, and $G$ are on sides $\overline{AB}$, $\overline{BC}$ and $\overline{DA}$, respectively, so that lengths $\overline{BE}$, $\overline{BF}$, and $\overline{DG}$ are equal. Points $H$ and $I$ are the midpoints of segments $\overline{EF}$ and $\overline{CG}$, respectively. Segment $\overline{GJ}$ is the perpendicular bisector of segment $\overline{HI}$. The ratio of the areas of pentagon $AEHJG$ and quadrilateral $CIHF$ can be written as $\dfrac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

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**30** Let $x$ and $y$ be real numbers satisfying

$$(x^2 + x - 1)(x^2 - x + 1) = 2(y^3 - 2\sqrt{5} - 1)$$

and

$$(y^2 + y - 1)(y^2 - y + 1) = 2(x^3 + 2\sqrt{5} - 1)$$

Find $8x^2 + 4y^3$.

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