Rioplatense Mathematical Olympiad, Level 3 2003

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by Shu

Day 1 December 10th

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1 Let x, y, and z be positive real numbers satisfying $x^2+y^2+z^2=1$. Prove that

$$x^2yz+xy^2z+xyz^2\le\frac{1}{3}.$$

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2 Triangle ABC is inscribed in the circle Γ. Let Γ_a denote the circle internally tangent to Γ and also tangent to sides AB and AC. Let A' denote the point of tangency of Γ and Γ_a. Define B' and C' similarly. Prove that AA', BB' and CC' are concurrent.

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3 An 8×8 chessboard is to be tiled (i.e., completely covered without overlapping) with pieces of the following shapes:

1×3 rectangle   T-shaped tetromino

The 1×3 rectangle covers exactly three squares of the chessboard, and the T-shaped tetromino covers exactly four squares of the chessboard. (a) What is the maximum number of pieces that can be used?
(b) How many ways are there to tile the chessboard using this maximum number of pieces?

Day 2 December 11th

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1 Inside right angle XAY, where A is the vertex, is a semicircle Γ whose center lies on AX and that is tangent to AY at the point A. Describe a ruler-and-compass construction for the tangent to Γ such that the triangle enclosed by the tangent and angle XAY has minimum area.

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2 Let n and k be positive integers. Consider n infinite arithmetic progressions of nonnegative integers with the property that among any k consecutive nonnegative integers, at least one of k integers belongs to one of the n arithmetic progressions. Let $d_1,d_2,\dots,d_n$ denote the differences of the arithmetic progressions, and let $d=\min\{d_1,d_2,\dots,d_n\}$. In terms of n and k, what is the maximum possible value of d?

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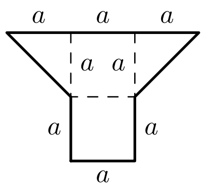
3 Without overlapping, hexagonal tiles are placed inside an isosceles right triangle of area 1 whose hypotenuse is horizontal. The tiles are similar to the figure below, but are not necessarily all the same size.



The longest side of each tile is parallel to the hypotenuse of the triangle, and the horizontal side of length a of each tile lies between this longest side of the tile and the hypotenuse of the triangle. Furthermore, if the longest side of a tile is farther from the hypotenuse than the longest side of another tile, then the size of the first tile is larger or equal to the size of the second tile. Find the smallest value of λ such that every such configuration of tiles has a total area less than λ.