Rioplatense Mathematical Olympiad, Level 3 2004

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by Shu

Day 1

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1 Find all polynomials $P(x)$ with real coefficients such that

$$xP\left(\frac{y}{x}\right)+yP\left(\frac{x}{y}\right)=x+y$$

for all nonzero real numbers x and y.

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2 Find the smallest integer n such that each subset of $\{1,2,\ldots,2004\}$ with n elements has two distinct elements a and b for which a^2-b^2 is a multiple of 2004.

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3 In a convex hexagon $ABCDEF$, triangles ACE and BDF have the same circumradius R. If triangle ACE has inradius r, prove that

$$\text{Area}(ABCDEF)\le\frac{R}{r}\cdot\text{Area}(ACE).$$

Day 2

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1 How many integers $n>1$ are there such that n divides $x^{13}-x$ for every positive integer x?

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2 A collection of cardboard circles, each with a diameter of at most 1, lie on a 5×8 table without overlapping or overhanging the edge of the table. A cardboard circle of diameter 2 is added to the collection. Prove that this new collection of cardboard circles can be placed on a 7×7 table without overlapping or overhanging the edge.

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3 Consider a partition of $\{1,2,\ldots,900\}$ into 30 subsets $S_1,S_2,\ldots,S_{30}$ each with 30 elements. In each S_k, we paint the fifth largest number blue. Is it possible that, for $k=1,2,\ldots,30$, the sum of the elements of S_k exceeds the sum of the blue numbers?
