Rioplatense Mathematical Olympiad, Level 3 2006
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Day 1

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1 (a) For each integer $k \geq 3$, find a positive integer n that can be represented as the sum of exactly k mutually distinct positive divisors of n.
(b) Suppose that n can be expressed as the sum of exactly k mutually distinct positive divisors of n for some $k \geq 3$. Let p be the smallest prime divisor of n. Show that

$$\frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{p+k-1} \geq 1.$$

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2 Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $CB = CD$. The bisector of $\angle BDC$ intersects BC at L, and AL intersects BD at M, and it is known that $BL = BM$. Determine the value of $2\angle BAD + 3\angle BCD$.

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3 The numbers $1, 2, \ldots, 2006$ are written around the circumference of a circle. A *move* consists of exchanging two adjacent numbers. After a sequence of such moves, each number ends up 13 positions to the right of its initial position. If the numbers $1, 2, \ldots, 2006$ are partitioned into 1003 distinct pairs, then show that in at least one of the moves, the two numbers of one of the pairs were exchanged.

Day 2

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1 The acute triangle ABC with $AB \neq AC$ has circumcircle Γ, circumcenter O, and orthocenter H. The midpoint of BC is M, and the extension of the median AM intersects Γ at N. The circle of diameter AM intersects Γ again at A and P. Show that the lines AP, BC, and OH are concurrent if and only if $AH = HN$.

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2 A given finite number of lines in the plane, no two of which are parallel and no three of which are concurrent, divide the plane into finite and infinite regions. In each finite region we write 1 or -1. In one operation, we can choose any triangle made of three of the lines (which may be cut by other lines in the collection) and multiply by -1 each of the numbers in the triangle. Determine if it is always possible to obtain 1 in all the finite regions by successively applying this operation, regardless of the initial distribution of 1s and -1s.

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3 An infinite sequence $x_1, x_2, \ldots$ of positive integers satisfies

$$x_{n+2} = \gcd(x_{n+1}, x_n) + 2006$$

for each positive integer n. Does there exist such a sequence which contains exactly 10^{2006} distinct numbers?