Rioplatense Mathematical Olympiad, Level 3 2009

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by Shu

Day 1 December 7th

1 Find all pairs (a, b) of real numbers with the following property:
Given any real numbers c and d, if both of the equations $x^2 + ax + 1 = c$ and $x^2 + bx + 1 = d$ have real roots, then the equation $x^2 + (a + b)x + 1 = cd$ has real roots.

2 Let A, B, C, D, E, F, G, H, I be nine points in space such that $ABCDE$, $ABFGH$, and $GFCDI$ are each regular pentagons with side length 1. Determine the lengths of the sides of triangle EHI.

3 Call a permutation of the integers $(1, 2, \ldots, n)$ [i]d-ordered[/i] if it does not contains a decreasing subsequence of length d. Prove that for every $d = 2, 3, \ldots, n$, the number of d-ordered permutations of $(1, 2, \ldots, n)$ is at most $(d - 1)^{2n}$.

Day 2 December 8th

1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy) = \max\{f(x + y), f(x)f(y)\}$$

for all real numbers x and y.

2 Find all pairs (a, b) of integers with $a > 1$ and $b > 1$ such that a divides $b + 1$ and b divides $a^3 - 1$.

3 Alice and Bob play the following game. It begins with a set of 1000 1×2 rectangles. A *move* consists of choosing two rectangles (a rectangle may consist of one or several 1×2 rectangles combined together) that share a common side length and combining those two rectangles into one rectangle along those sides sharing that common length. The first player who cannot make a move loses. Alice moves first. Describe a winning strategy for Bob.