Rioplatense Mathematical Olympiad, Level 3 2010

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Day 1 December 7th

1 Suppose a, b, c, and d are distinct positive integers such that a^b divides b^c, b^c divides c^d, and c^d divides d^a.
(a) Is it possible to determine which of the numbers a, b, c, d is the smallest?
(b) Is it possible to determine which of the numbers a, b, c, d is the largest?

2 Acute triangle ABP, where $AB > BP$, has altitudes BH, PQ, and AS. Let C denote the intersection of lines QS and AP, and let L denote the intersection of lines HS and BC. If $HS = SL$ and HL is perpendicular to BC, find the value of $\frac{SL}{SC}$.

3 Find all the functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{N}$ satisfying $10^6 - \frac{1}{10^6} < \frac{x}{y} < 10^6 + \frac{1}{10^6}$.

Note: $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}$ denotes the set of real numbers.

Day 2 December 8th

1 Let $r_2, r_3, \ldots, r_{1000}$ denote the remainders when a positive odd integer is divided by $2, 3, \ldots, 1000$, respectively. It is known that the remainders are pairwise distinct and one of them is 0. Find all values of k for which it is possible that $r_k = 0$.

2 Find the minimum and maximum values of $S = \frac{a}{b} + \frac{c}{d}$ where a, b, c, d are positive integers satisfying $a + c = 20202$ and $b + d = 20200$.

3 Alice and Bob play the following game. To start, Alice arranges the numbers $1, 2, \ldots, n$ in some order in a row and then Bob chooses one of the numbers and places a pebble on it. A player's *turn* consists of picking up and placing the pebble on an adjacent number under the restriction that the pebble can be placed on the number k at most k times. The two players alternate taking turns beginning with Alice. The first player who cannot make a move loses. For each positive integer n, determine who has a winning strategy.