Rioplatense Mathematical Olympiad, Level 3 2013

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Day 1

1 Let a, b, c, d be real positive numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. Prove that $(1-a)(1-b)(1-c)(1-d) \geq abcd$.

2 Let $ABCD$ be a square, and let E and F be points in AB and BC respectively such that $BE = BF$. In the triangle EBC, let N be the foot of the altitude relative to EC. Let G be the intersection between AD and the extension of the previously mentioned altitude. FG and EC intersect at point P, and the lines NF and DC intersect at point T. Prove that the line DP is perpendicular to the line BT.

3 A division of a group of people into various groups is called k-regular if the number of groups is less or equal to k and two people that know each other are in different groups.
Let A, B, and C groups of people such that there are is no person in A and no person in B that know each other. Suppose that the group $A \cup C$ has an a-regular division and the group $B \cup C$ has a b-regular division.
For each a and b, determine the least possible value of k for which it is guaranteed that the group $A \cup B \cup C$ has a k-regular division.

Day 2

4 Two players A and B play alternatively in a convex polygon with $n \geq 5$ sides. In each turn, the corresponding player has to draw a diagonal that does not cut inside the polygon previously drawn diagonals. A player loses if after his turn, one quadrilateral is formed such that its two diagonals are not drawn. A starts the game.
For each positive integer n, find a winning strategy for one of the players.

5 Find all positive integers n for which there exist two distinct numbers of n digits, $\overline{a_1a_2\ldots a_n}$ and $\overline{b_1b_2\ldots b_n}$, such that the number of $2n$ digits $\overline{a_1a_2\ldots a_nb_1b_2\ldots b_n}$ is divisible by $\overline{b_1b_2\ldots b_na_1a_2\ldots a_n}$.

6 Let ABC be an acute scalene triangle, H its orthocenter and G its geocenter. The circumference with diameter AH cuts the circumcircle of BHC in A' ($A' \neq H$). Points B' and C' are defined similarly. Show that the points A', B', C', and G lie in one circumference.