**Saint Petersburg Mathematical Olympiad 2008**

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–      Grade 9 Problems

**1**      The graph $y = x^2 + ax + b$ intersects any of the two axes at points $A$, $B$, and $C$. The incenter of triangle $ABC$ lies on the line $y = x$. Prove that $a + b + 1 = 0$.

**2**      In a kingdom, there are roads open between some cities with lanes both ways, in such a way, that you can come from one city to another using those roads. The roads are toll, and the price for taking each road is distinct. A minister made a list of all routes that go through each city exactly once. The king marked the most expensive road in each of the routes and said to close all the roads that he marked at least once. After that, it became impossible to go from city $A$ to city $B$, from city $B$ to city $C$, and from city $C$ to city $A$. Prove that the kings order was followed incorrectly.

**3**      Pentagon $ABCDE$ has circle $S$ inscribed into it. Side $BC$ is tangent to $S$ at point $K$. If $AB = BC = CD$, prove that angle $EKB$ is a right angle.

**4**      The numbers $x_1, ... x_{100}$ are written on a board so that $x_1 = \frac{1}{2}$ and for every $n$ from 1 to 99, $x_{n+1} = 1 - x_1 x_2 x_3 * ... * x_{100}$. Prove that $x_{100} > 0.99$.

**5**      Given are distinct natural numbers $a$, $b$, and $c$. Prove that

$$\gcd(ab + 1, ac + 1, bc + 1) \leq \frac{a + b + c}{3}$$

**6**      In cyclic quadrilateral $ABCD$ rays $AB$ and $DC$ intersect at point $E$, while segments $AC$ and $BD$ intersect at $F$. Point $P$ is on ray $EF$ such that angles $BPE$ and $CPE$ are congruent. Prove that angles $APB$ and $DPC$ are also equal.

**7**      A square with side $2008$ is broken into regions that are all squares with side $1$. In every region, either $0$ or $1$ is written, and the number of $1$'s and $0$'s is the same. The border between two of the regions is removed, and the numbers in each of them are also removed, while in the new region, their arithmetic mean is recorded. After several of those operations, there is only one square left, which is the big square itself. Prove that it is possible to perform these operations in such a way, that the final number in the big square is less than $\frac{1}{2^{10^6}}$.

–      Grade 10 Problems

**1** Replacing any of the coefficients of quadratic trinomial $f(x) = ax^2 + bx + c$ with an $1$ will result in a quadratic trinomial with at least one real root. Prove that the resulting trinomial attains a negative value at at least one point.

EDIT: Oops I failed, added "with a 1." Also, I am sorry for not knowing these are posted already, however, these weren't posted in the contest lab yet, which made me think they weren't translated yet.

Note: fresh translation

**2** Point $O$ is the center of the circle into which quadrilateral $ABCD$ is inscribed. If angles $AOC$ and $BAD$ are both equal to $110$ degrees and angle $ABC$ is greater than angle $ADC$, prove that $AB + AD > CD$.

Fresh translation.

**3** Same as G9P3

**4** A wizard thinks of a number from $1$ to $n$. You can ask the wizard any number of yes/no questions about the number. The wizard must answer all those questions, but not necessarily in the respective order. What is the least number of questions that must be asked in order to know what the number is for sure. (In terms of $n$.)

Fresh translation.

**5** Same as G9P6

**6** A diagonal of a 100-gon is called good if it divides the 100-gon into two polygons each with an odd number of sides. A 100-gon was split into triangles with non-intersecting diagonals, exactly 49 of which are good. The triangles are colored into two colors such that no two triangles that border each other are colored with the same color. Prove that there is the same number of triangles colored with one color as with the other.

Fresh translation; slightly reworded.

**7** In a sequence, $x_1 = \frac{1}{2}$ and $x_{n+1} = 1 - x_1 x_2 x_3 ... x_n$ for $n \geq 1$. Prove that $0.99 < x_{100} < 0.991$.

Fresh translation. This problem may be similar to one of the 9th grade problems.

**–** Grade 11 Problems

**1** We color some cells in $10000 \times 10000$ square, such that every $10 \times 10$ square and every $1 \times 100$ line have at least one coloring cell. What minimum number of cells we should color ?

**2** Same as G9P3

**3**    There are $2008$ trinomials $x^2 - a_k x + b_k$ where $a_k$ and $b_k$ are all different numbers from set $(1, 2, ..., 4016)$. These trinomials has not common real roots. We mark all real roots on the $Ox$-axis.

Prove, that distance between some two marked points is $\leq \frac{1}{250}$

**4**    There are $100$ numbers on circle, and no one number is divided by other. In same time for all numbers we make next operation:

If $(a, b)$ are two neighbors ($a$ is left neighbor) , then we write between $a, b$ number $\frac{a}{(a,b)}$ and erase $a, b$

This operation was repeated some times. What maximum number of $1$ we can receive ?

Example: If we have circle with $3$ numbers $4, 5, 6$ then after operation we receive circle with numbers $\frac{4}{(4,5)} = 4, \frac{5}{(5,6)} = 5, \frac{6}{(6,4)} = 3$.

**5**    All faces of the tetrahedron $ABCD$ are acute-angled triangles. $AK$ and $AL$ -are altitudes in faces $ABC$ and $ABD$. Points $C, D, K, L$ lies on circle. Prove, that $AB \perp CD$

**6**    $a + b + c \leq 3000000$ and $a \neq b \neq c \neq a$ and $a, b, c$ are naturals.
Find maximum $GCD(ab + 1, ac + 1, bc + 1)$

**7**    There are $10000$ cities in country, and roads between some cities. Every city has $< 100$ roads. Every cycle route with odd number of road consists of $\geq 101$ roads.
Prove that we can divide all cities in $100$ groups with $100$ cities, such that every road leads from one group to other.