9th RMM 2017

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– Day 1 (February 24, 2017)

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1 **(a)** Prove that every positive integer n can be written uniquely in the form

$$n=\sum_{j=1}^{2k+1}(-1)^{j-1}2^{m_j},$$

where $k\geq 0$ and $0\leq m_1<m_2\cdots<m_{2k+1}$ are integers.
This number k is called *weight* of n.

(b) Find (in closed form) the difference between the number of positive integers at most 2^{2017} with even weight and the number of positive integers at most 2^{2017} with odd weight.

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2 Determine all positive integers n satisfying the following condition: for every monic polynomial P of degree at most n with integer coefficients, there exists a positive integer $k\leq n$ and $k+1$ distinct integers $x_1,x_2,\cdots,x_{k+1}$ such that

$$P(x_1)+P(x_2)+\cdots+P(x_k)=P(x_{k+1})$$

.

Note. A polynomial is *monic* if the coefficient of the highest power is one.

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3 Let n be an integer greater than 1 and let X be an n-element set. A non-empty collection of subsets $A_1,...,A_k$ of X is tight if the union $A_1\cup\cdots\cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_is. Find the largest cardinality of a collection of proper non-empty subsets of X, no non-empty subcollection of which is tight.

Note. A subset A of X is proper if $A\neq X$. The sets in a collection are assumed to be distinct. The whole collection is assumed to be a subcollection.

– Day 2 (February 25, 2017)

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4 In the Cartesian plane, let G_1 and G_2 be the graphs of the quadratic functions $f_1(x)=p_1x^2+q_1x+r_1$ and $f_2(x)=p_2x^2+q_2x+r_2$, where $p_1>0>p_2$. The graphs G_1 and G_2 cross at distinct points A and B. The four tangents to G_1 and G_2 at A and B form a convex quadrilateral which has an inscribed circle. Prove that the graphs G_1 and G_2 have the same axis of symmetry.

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5 Fix an integer $n \geq 2$. An $n \times n$ sieve is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A stick is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

Palmer Mebane

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6 Let $ABCD$ be any convex quadrilateral and let P, Q, R, S be points on the segments AB, BC, CD, and DA, respectively. It is given that the segments PR and QS dissect $ABCD$ into four quadrilaterals, each of which has perpendicular diagonals. Show that the points P, Q, R, S are concyclic.
