Turkey Junior National Olympiad 2012

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1 Let x, y be integers and p be a prime for which

$$x^2 - 3xy + p^2y^2 = 12p$$

Find all triples (x, y, p).

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2 In a convex quadrilateral $ABCD$, the diagonals are perpendicular to each other and they intersect at E. Let P be a point on the side AD which is different from A such that $PE = EC$. The circumcircle of triangle BCD intersects the side AD at Q where Q is also different from A. The circle, passing through A and tangent to line EP at P, intersects the line segment AC at R. If the points B, R, Q are concurrent then show that $\angle BCD = 90^\circ$.

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3 Let a, b, c be positive real numbers satisfying $a^3 + b^3 + c^3 = a^4 + b^4 + c^4$. Show that

$$\frac{a}{a^2 + b^3 + c^3} + \frac{b}{a^3 + b^2 + c^3} + \frac{c}{a^3 + b^3 + c^2} \geq 1$$

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4 We want to place 2012 pockets, including variously colored balls, into k boxes such that

i) For any box, all pockets in this box must include a ball with the same color
or
ii) For any box, all pockets in this box must include a ball having a color which is not included in any other pocket in this box

Find the smallest value of k for which we can always do this placement whatever the number of balls in the pockets and whatever the colors of balls.
