Mexico National Olympiad 2008

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Day 1

1 Let $1 = d_1 < d_2 < d_3 < \cdots < d_k = n$ be the divisors of n. Find all values of n such that $n = d_2^2 + d_3^3$.

2 Consider a circle Γ, a point A on its exterior, and the points of tangency B and C from A to Γ. Let P be a point on the segment AB, distinct from A and B, and let Q be the point on AC such that PQ is tangent to Γ. Points R and S are on lines AB and AC, respectively, such that $PQ \parallel RS$ and RS is tangent to Γ as well. Prove that $[APQ] \cdot [ARS]$ does not depend on the placement of point P.

3 Consider a chess board, with the numbers 1 through 64 placed in the squares as in the diagram below.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Assume we have an infinite supply of knights. We place knights in the chess board squares such that no two knights attack one another and compute the sum of the numbers of the cells on which the knights are placed. What is the maximum sum that we can attain?

Note. For any 2×3 or 3×2 rectangle that has the knight in its corner square, the knight can attack the square in the opposite corner.

Day 2

1 A king decides to reward one of his knights by making a game. He sits the knights at a round table and has them call out $1, 2, 3, 1, 2, 3, \ldots$ around the circle (that is, clockwise, and each person says a number). The people who say 2 or 3 immediately lose, and this continues until the last knight is left, the winner.

Numbering the knights initially as $1, 2, \dots, n$, find all values of n such that knight 2008 is the winner.

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2 We place 8 distinct integers in the vertices of a cube and then write the greatest common divisor of each pair of adjacent vertices on the edge connecting them. Let E be the sum of the numbers on the edges and V the sum of the numbers on the vertices.

a) Prove that $\frac{2}{3}E \leq V$.
b) Can $E = V$?

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3 The internal angle bisectors of A, B, and C in $\triangle ABC$ concur at I and intersect the circumcircle of $\triangle ABC$ at L, M, and N, respectively. The circle with diameter IL intersects BC at D and E; the circle with diameter IM intersects CA at F and G; the circle with diameter IN intersects AB at H and J. Show that D, E, F, G, H, and J are concyclic.
