Mexico National Olympiad 2009

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Day 1

1 In $\triangle ABC$, let D be the foot of the altitude from A to BC. A circle centered at D with radius AD intersects lines AB and AC at P and Q, respectively. Show that $\triangle AQP \sim \triangle ABC$.

2 In boxes labeled $0, 1, 2, \dots$, we place integers according to the following rules:

- If p is a prime number, we place it in box 1.
- If a is placed in box m_a and b is placed in box m_b, then ab is placed in the box labeled am_b+bm_a.

Find all positive integers n that are placed in the box labeled n.

3 Let a, b, and c be positive numbers satisfying $abc=1$. Show that

$$\frac{a^3}{a^3+2}+\frac{b^3}{b^3+2}+\frac{c^3}{c^3+2}\ge 1 \text{ and } \frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2}\le 1$$

Day 2

1 Let $n>1$ be an odd integer, and let a_1, a_2, $\dots$, a_n be distinct real numbers. Let M be the maximum of these numbers and m the minimum. Show that it is possible to choose the signs of the expression $s=\pm a_1\pm a_2\pm\cdots\pm a_n$ so that

$$m<s<M$$

2 Consider a triangle ABC and a point M on side BC. Let P be the intersection of the perpendiculars from M to AB and from B to BC, and let Q be the intersection of the perpendiculars from M to AC and from C to BC. Show that PQ is perpendicular to AM if and only if M is the midpoint of BC.

3 At a party with n people, it is known that among any 4 people, there are either 3 people who all know one another or 3 people none of which knows another. Show that the n people can be separated into two rooms, so that everyone in one room knows one another and no two people in the other room know each other.