Mexico National Olympiad 2010

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Day 1

1 Find all triplets of natural numbers (a, b, c) that satisfy the equation $abc = a + b + c + 1$.

2 In each cell of an $n \times n$ board is a lightbulb. Initially, all of the lights are off. Each move consists of changing the state of all of the lights in a row or of all of the lights in a column (off lights are turned on and on lights are turned off).

Show that if after a certain number of moves, at least one light is on, then at this moment at least n lights are on.

3 Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be externally tangent at a point A. A line tangent to $\mathcal{C}_1$ at B intersects $\mathcal{C}_2$ at C and D; then the segment AB is extended to intersect $\mathcal{C}_2$ at a point E. Let F be the midpoint of $\overset{\frown}{CD}$ that does not contain E, and let H be the intersection of BF with $\mathcal{C}_2$. Show that CD, AF, and EH are concurrent.

Day 2

1 Let n be a positive integer. In an $n \times 4$ table, each row is equal to

2	0	1	0

A *change* is taking three consecutive boxes in the same row with different digits in them and changing the digits in these boxes as follows:

$$0 \to 1, 1 \to 2, 2 \to 0.$$

For example, a row $\boxed{2}\boxed{0}\boxed{1}\boxed{0}$ can be changed to the row $\boxed{0}\boxed{1}\boxed{2}\boxed{0}$ but not to $\boxed{2}\boxed{1}\boxed{2}\boxed{1}$ because 0, 1, and 0 are not distinct.

Changes can be applied as often as wanted, even to items already changed. Show that for $n < 12$, it is not possible to perform a finite number of changes so that the sum of the elements in each column is equal.

2 Let ABC be an acute triangle with $AB \neq AC$, M be the median of BC, and H be the orthocenter of $\triangle ABC$. The circumcircle of B, H, and C intersects the median AM at N. Show that $\angle ANH = 90^\circ$.

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3 Let p, q, and r be distinct positive prime numbers. Show that if

$$pqr \mid (pq)^r + (qr)^p + (rp)^q - 1,$$

then

$$(pqr)^3 \mid 3((pq)^r + (qr)^p + (rp)^q - 1).$$
