Mexico National Olympiad 2011

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Day 1

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1 25 lightbulbs are distributed in the following way: the first 24 are placed on a circumference, placing a bulb at each vertex of a regular 24-gon, and the remaining bulb is placed on the center of said circumference.
At any time, the following operations may be applied:

- Take two vertices on the circumference with an odd amount of vertices between them, and change the state of the bulbs on those vertices and the center bulb.
- Take three vertices on the circumference that form an equilateral triangle, change the state of the bulbs on those vertices and the center bulb.

Prove from any starting configuration of on and off lightbulbs, it is always possible to reach a configuration where all the bulbs are on.

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2 Let ABC be an acute triangle and Γ its circumcircle. Let l be the line tangent to Γ at A. Let D and E be the intersections of the circumference with center B and radius AB with lines l and AC, respectively. Prove the orthocenter of ABC lies on line DE.

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3 Let n be a positive integer. Find all real solutions $(a_1, a_2, \dots, a_n)$ to the system:

$$a_1^2 + a_1 - 1 = a_2$$
$$a_2^2 + a_2 - 1 = a_3$$
$$\vdots$$
$$a_n^2 + a_n - 1 = a_1$$

Day 2

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4 Find the smallest positive integer that uses exactly two different digits when written in decimal notation and is divisible by all the numbers from 1 to 9.

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5 A $(2^n - 1) \times (2^n + 1)$ board is to be divided into rectangles with sides parallel to the sides of the board and integer side lengths such that the area of each rectangle is a power of 2. Find the minimum number of rectangles that the board may be divided into.

6 Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two circumferences intersecting at points A and B. Let C be a point on line AB such that B lies between A and C. Let P and Q be points on $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively such that CP and CQ are tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, P is not inside $\mathcal{C}_2$ and Q is not inside $\mathcal{C}_1$. Line PQ cuts $\mathcal{C}_1$ at R and $\mathcal{C}_2$ at S, both points different from P, Q and B. Suppose CR cuts $\mathcal{C}_1$ again at X and CS cuts $\mathcal{C}_2$ again at Y. Let Z be a point on line XY. Prove SZ is parallel to QX if and only if PZ is parallel to RX.