**Mexico National Olympiad 2012**

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by randomusername

**Day 1**

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**1** Let $\mathcal{C}_1$ be a circumference with center $O$, $P$ a point on it and $\ell$ the line tangent to $\mathcal{C}_1$ at $P$. Consider a point $Q$ on $\ell$ different from $P$, and let $\mathcal{C}_2$ be the circumference passing through $O$, $P$ and $Q$. Segment $OQ$ cuts $\mathcal{C}_1$ at $S$ and line $PS$ cuts $\mathcal{C}_2$ at a point $R$ diffferent from $P$. If $r_1$ and $r_2$ are the radii of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, Prove

$$\frac{PS}{SR}=\frac{r_1}{r_2}.$$

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**2** Let $n \geq 4$ be an even integer. Consider an $n\times n$ grid. Two cells ($1\times 1$ squares) are *neighbors* if they share a side, are in opposite ends of a row, or are in opposite ends of a column. In this way, each cell in the grid has exactly four neighbors.
An integer from 1 to 4 is written inside each square according to the following rules:

-If a cell has a 2 written on it, then at least two of its neighbors contain a 1.
-If a cell has a 3 written on it, then at least three of its neighbors contain a 1.
-If a cell has a 4 written on it, then all of its neighbors contain a 1.

Among all arrangements satisfying these conditions, what is the maximum number that can be obtained by adding all of the numbers on the grid?

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**3** Prove among any $14$ consecutive positive integers there exist $6$ which are pairwise relatively prime.

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**Day 2**

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**4** The following process is applied to each positive integer: the sum of its digits is subtracted from the number, and the result is divided by $9$. For example, the result of the process applied to $938$ is $102$, since $\frac{938-(9+3+8)}{9}=102$. Applying the process twice to $938$ the result is $11$, applied three times the result is $1$, and applying it four times the result is $0$. When the process is applied one or more times to an integer $n$, the result is eventually $0$. The number obtained before obtaining $0$ is called the *house* of $n$.

How many integers less than $26000$ share the same *house* as $2012$?

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**5** Some frogs, some red and some others green, are going to move in an $11\times 11$ grid, according to the following rules. If a frog is located, say, on the square marked with# in the following diagram, then

-If it is red, it can jump to any square marked with an x.
-if it is green, it can jump to any square marked with an o.

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | x | | o | | |
| | o | | | | x | |
| | | | # | | | |
| | x | | | | o | |
| | | o | | x | | |
| | | | | | | |

We say 2 frogs (of any color) can meet at a square if both can get to the same square in one or more jumps, not neccesarily with the same amount of jumps.

-Prove if 6 frogs are placed, then there exist at least 2 that can meet at a square.
-For which values of $k$ is it possible to place one green and one red frog such that they can meet at exactly $k$ squares?

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**6** Consider an acute triangle $ABC$ with circumcircle $\mathcal{C}$. Let $H$ be the orthocenter of $ABC$ and $M$ the midpoint of $BC$. Lines $AH$, $BH$ and $CH$ cut $\mathcal{C}$ again at points $D$, $E$, and $F$ respectively; line $MH$ cuts $\mathcal{C}$ at $J$ such that $H$ lies between $J$ and $M$. Let $K$ and $L$ be the incenters of triangles $DEJ$ and $DFJ$ respectively. Prove $KL$ is parallel to $BC$.

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