Mexico National Olympiad 2013

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1 All the prime numbers are written in order, $p_1 = 2, p_2 = 3, p_3 = 5, ...$
Find all pairs of positive integers a and b with $a - b \geq 2$, such that $p_a - p_b$ divides $2(a-b)$.

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2 Let $ABCD$ be a parallelogram with the angle at A obtuse. Let P be a point on segment BD. The circle with center P passing through A cuts line AD at A and Y and cuts line AB at A and X. Line AP intersects BC at Q and CD at R. Prove $\angle XPY = \angle XQY + \angle XRY$.

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3 What is the largest amount of elements that can be taken from the set $\{1, 2, ..., 2012, 2013\}$, such that within them there are no distinct three, say a, b,and c, such that a is a divisor or multiple of $b - c$?

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4 A $n \times n \times n$ cube is constructed using $1 \times 1 \times 1$ cubes, some of them black and others white, such that in each $n \times 1 \times 1$, $1 \times n \times 1$, and $1 \times 1 \times n$ subprism there are exactly two black cubes, and they are separated by an even number of white cubes (possibly 0).
Show it is possible to replace half of the black cubes with white cubes such that each $n \times 1 \times 1$, $1 \times n \times 1$ and $1 \times 1 \times n$ subprism contains exactly one black cube.

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5 A pair of integers is special if it is of the form $(n, n-1)$ or $(n-1, n)$ for some positive integer n. Let n and m be positive integers such that pair (n, m) is not special. Show (n, m) can be expressed as a sum of two or more different special pairs if and only if n and m satisfy the inequality $n + m \geq (n-m)^2$.
Note: The sum of two pairs is defined as $(a, b) + (c, d) = (a + c, b + d)$.

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6 Let $A_1A_2...A_8$ be a convex octagon such that all of its sides are equal and its opposite sides are parallel. For each $i = 1, ..., 8$, define B_i as the intersection between segments A_iA_{i+4} and $A_{i-1}A_{i+1}$, where $A_{j+8} = A_j$ and $B_{j+8} = B_j$ for all j. Show some number i, amongst 1, 2, 3, and 4 satisfies

$$\frac{A_iA_{i+4}}{B_iB_{i+4}} \leq \frac{3}{2}$$
