Mexico National Olympiad 2014

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Day 1

1 Each of the integers from 1 to 4027 has been colored either green or red. Changing the color of a number is making it red if it was green and making it green if it was red. Two positive integers m and n are said to be *cuates* if either $\frac{m}{n}$ or $\frac{n}{m}$ is a prime number. A *step* consists in choosing two numbers that are cuates and changing the color of each of them. Show it is possible to apply a sequence of steps such that every integer from 1 to 2014 is green.

2 A positive integer a is said to *reduce* to a positive integer b if when dividing a by its units digits the result is b. For example, 2015 reduces to $\frac{2015}{5} = 403$.
Find all the positive integers that become 1 after some amount of reductions. For example, 12 is one such number because 12 reduces to 6 and 6 reduces to 1.

3 Let Γ_1 be a circle and P a point outside of Γ_1. The tangents from P to Γ_1 touch the circle at A and B. Let M be the midpoint of PA and Γ_2 the circle through P, A and B. Line BM cuts Γ_2 at C, line CA cuts Γ_1 at D, segment DB cuts Γ_2 at E and line PE cuts Γ_1 at F, with E in segment PF. Prove lines AF, BP, and CE are concurrent.

Day 2

4 Problem 4

Let $ABCD$ be a rectangle with diagonals AC and BD. Let E be the intersection of the bisector of $\angle CAD$ with segment CD, F on CD such that E is midpoint of DF, and G on BC such that $BG = AC$ (with C between B and G). Prove that the circumference through D, F and G is tangent to BG.

5 Let a, b, c be positive reals such that $a + b + c = 3$. Prove:

$$\frac{a^2}{a+\sqrt[3]{bc}} + \frac{b^2}{b+\sqrt[3]{ca}} + \frac{c^2}{c+\sqrt[3]{ab}} \geq \frac{3}{2}$$

And determine when equality holds.

6 Let $d(n)$ be the number of positive divisors of a positive integer n (including 1 and n). Find all values of n such that $n + d(n) = d(n)^2$.