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Day 1

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1 Find all infinite sequences $a_1, a_2, \ldots$ of positive integers satisfying the following properties:
(a) $a_1 < a_2 < a_3 < \cdots$,
(b) there are no positive integers i, j, k, not necessarily distinct, such that $a_i + a_j = a_k$,
(c) there are infinitely many k such that $a_k = 2k - 1$.

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2 Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and the circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $MN \parallel O_1O_2$.

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3 Let $\mathbb{N}$ be the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following two conditions:
(a) $f(m)$ and $f(n)$ are relatively prime whenever m and n are relatively prime.
(b) $n \le f(n) \le n + 2012$ for all n.

Prove that for any natural number n and any prime p, if p divides $f(n)$ then p divides n.

Day 2

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4 In scalene triangle ABC, let the feet of the perpendiculars from A to BC, B to CA, C to AB be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides BC, CA, AB. Show that the perpendiculars from D to AA_2, E to BB_2 and F to CC_2 are concurrent.

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5 A rational number x is given. Prove that there exists a sequence $x_0, x_1, x_2, \ldots$ of rational numbers with the following properties:
(a) $x_0 = x$;
(b) for every $n \ge 1$, either $x_n = 2x_{n-1}$ or $x_n = 2x_{n-1} + \frac{1}{n}$;
(c) x_n is an integer for some n.

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6 Positive real numbers x, y, z satisfy $xyz + xy + yz + zx = x + y + z + 1$. Prove that

$$\frac{1}{3}\left(\sqrt{\frac{1+x^2}{1+x}} + \sqrt{\frac{1+y^2}{1+y}} + \sqrt{\frac{1+z^2}{1+z}}\right) \le \left(\frac{x+y+z}{3}\right)^{5/8}.$$

Day 3

7 Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

8 Let n be a positive integer. Consider a triangular array of nonnegative integers as follows:

$$\begin{array}{rccccccccc}
\text{Row } 1: & & & & & a_{0,1} & & & & \\
\text{Row } 2: & & & & a_{0,2} & & a_{1,2} & & & \\
 & & & \vdots & & \vdots & & \vdots & & \\
\text{Row } n-1: & & a_{0,n-1} & & a_{1,n-1} & & \cdots & & a_{n-2,n-1} & \\
\text{Row } n: & a_{0,n} & & a_{1,n} & & a_{2,n} & & \cdots & & a_{n-1,n}
\end{array}$$

Call such a triangular array *stable* if for every $0 \le i < j < k \le n$ we have

$$a_{i,j} + a_{j,k} \le a_{i,k} \le a_{i,j} + a_{j,k} + 1.$$

For $s_1, \ldots s_n$ any nondecreasing sequence of nonnegative integers, prove that there exists a unique stable triangular array such that the sum of all of the entries in row k is equal to s_k.

9 Given a set S of n variables, a binary operation $\times$ on S is called *simple* if it satisfies $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in S$ and $x \times y \in \{x, y\}$ for all $x, y \in S$. Given a simple operation $\times$ on S, any string of elements in S can be reduced to a single element, such as $xyz \to x \times (y \times z)$. A string of variables in S is called *full* if it contains each variable in S at least once, and two strings are *equivalent* if they evaluate to the same variable regardless of which simple $\times$ is chosen. For example xxx, xx, and x are equivalent, but these are only full if $n = 1$. Suppose T is a set of strings such that any full string is equivalent to exactly one element of T. Determine the number of elements of T.

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