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Day 1 June 21st

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1 Let ABC be a triangle and D, E, F be the midpoints of arcs BC, CA, AB on the circumcircle. Line ℓ_a passes through the feet of the perpendiculars from A to DB and DC. Line m_a passes through the feet of the perpendiculars from D to AB and AC. Let A_1 denote the intersection of lines ℓ_a and m_a. Define points B_1 and C_1 similarly. Prove that triangle DEF and $A_1B_1C_1$ are similar to each other.

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2 A finite sequence of integers $a_1, a_2, \dots, a_n$ is called *regular* if there exists a real number x satisfying

$$\lfloor kx \rfloor = a_k \quad \text{for } 1 \le k \le n.$$

Given a regular sequence $a_1, a_2, \dots, a_n$, for $1 \le k \le n$ we say that the term a_k is *forced* if the following condition is satisfied: the sequence

$$a_1, a_2, \dots, a_{k-1}, b$$

is regular if and only if $b = a_k$. Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

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3 Divide the plane into an infinite square grid by drawing all the lines $x = m$ and $y = n$ for $m, n \in \mathbb{Z}$. Next, if a square's upper-right corner has both coordinates even, color it black; otherwise, color it white (in this way, exactly $1/4$ of the squares are black and no two black squares are adjacent). Let r and s be odd integers, and let (x, y) be a point in the interior of any white square such that $rx - sy$ is irrational. Shoot a laser out of this point with slope r/s; lasers pass through white squares and reflect off black squares. Prove that the path of this laser will form a closed loop.

Day 2 June 23rd

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4 Circle ω, centered at X, is internally tangent to circle Ω, centered at Y, at T. Let P and S be variable points on Ω and ω, respectively, such that line PS is tangent to ω (at S). Determine the locus of O – the circumcenter of triangle PST.

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5 Let p be a prime. Prove that any complete graph with $1000p$ vertices, whose edges are labelled with integers, has a cycle whose sum of labels is divisible by p.

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6 Let $\mathbb{N}$ be the set of positive integers. Find all functions $f: \mathbb{N} \to \mathbb{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \geq 2$.
(Here $f^1(n) = f(n)$ and $f^k(n) = f(f^{k-1}(n))$ for every integer k greater than 1.)

Day 3 June 25th

7 A country has n cities, labelled $1, 2, 3, \dots, n$. It wants to build exactly $n-1$ roads between certain pairs of cities so that every city is reachable from every other city via some sequence of roads. However, it is not permitted to put roads between pairs of cities that have labels differing by exactly 1, and it is also not permitted to put a road between cities 1 and n. Let T_n be the total number of possible ways to build these roads.
(a) For all odd n, prove that T_n is divisible by n.
(b) For all even n, prove that T_n is divisible by $n/2$.

8 Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$, $f(n+1) = f(n) + 2^{f(n)}$ for every positive integer n. Prove that $f(1), f(2), \dots, f(3^{2013})$ leave distinct remainders when divided by 3^{2013}.

9 Let r be a rational number in the interval $[-1, 1]$ and let $\theta = \cos^{-1} r$. Call a subset S of the plane *good* if S is unchanged upon rotation by θ around any point of S (in both clockwise and counterclockwise directions). Determine all values of r satisfying the following property: The midpoint of any two points in a good set also lies in the set.