USA TSTST 2014

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Day 1

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1 Let $\leftarrow$ denote the left arrow key on a standard keyboard. If one opens a text editor and types the keys "ab$\leftarrow$ cd $\leftarrow\leftarrow$ e $\leftarrow\leftarrow$ f", the result is "faecdb". We say that a string B is *reachable* from a string A if it is possible to insert some amount of $\leftarrow$'s in A, such that typing the resulting characters produces B. So, our example shows that "faecdb" is reachable from "abcdef".

Prove that for any two strings A and B, A is reachable from B if and only if B is reachable from A.

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2 Consider a convex pentagon circumscribed about a circle. We name the lines that connect vertices of the pentagon with the opposite points of tangency with the circle *gergonnians*.
(a) Prove that if four gergonnians are conncurrent, the all five of them are concurrent.
(b) Prove that if there is a triple of gergonnians that are concurrent, then there is another triple of gergonnians that are concurrent.

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3 Find all polynomials $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1 - x^2})$$

for all real x with $|x| \le 1$.

Day 2

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4 Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, and let d be the degree of $P(x)$. Assume that $P(x)$ is not the zero polynomial. Prove that there exist polynomials $A(x)$ and $B(x)$ such that:
(i) both A and B have degree at most $d/2$
(ii) at most one of A and B is the zero polynomial.
(iii) $\frac{A(x)+Q(x)B(x)}{P(x)}$ is a polynomial with real coefficients. That is, there is some polynomial $C(x)$ with real coefficients such that $A(x) + Q(x)B(x) = P(x)C(x)$.

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5 Find the maximum number E such that the following holds: there is an edge-colored graph with 60 vertices and E edges, with each edge colored either red or blue, such that in that coloring, there is no monochromatic cycles of length 3 and no monochromatic cycles of length 5.

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6 Suppose we have distinct positive integers a, b, c, d, and an odd prime p not dividing any of them, and an integer M such that if one considers the infinite sequence

$$\begin{gathered} ca - db \\ ca^2 - db^2 \\ ca^3 - db^3 \\ ca^4 - db^4 \\ \vdots \end{gathered}$$

and looks at the highest power of p that divides each of them, these powers are not all zero, and are all at most M. Prove that there exists some T (which may depend on a, b, c, d, p, M) such that whenever p divides an element of this sequence, the maximum power of p that divides that element is exactly p^T.

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