Bulgarian Winter Tournament 2025

www.artofproblemsolving.com/community/c4186913

by Assassino9931

10.2 Let D be an arbitrary point on the side BC of the non-isosceles acute triangle ABC. The circle with center D and radius DA intersects the rays $AB^{\to}$ (after B) and $AC^{\to}$ (after C) at M and N. Prove that the orthocenter of triangle AMN lies on a fixed line, independent of the choice of D.

10.3 In connection with the formation of a stable government, the President invited all 240 Members of Parliament to three separate consultations, where each member participated in exactly one consultation, and at each consultation there has been at least one member present. Discussions between pairs of members are to take place to discuss the consultations. Is it possible for these discussions to occur in such a way that there exists a non-negative integer k, such that for every two members who participated in different consultations, there are exactly k members who participated in the remaining consultation, with whom each of the two members has a conversation, and exactly k members who participated in the remaining consultation, with whom neither of the two has a conversation? If yes, then find all possible values of k.

10.4 The function $f : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is such that $f(a,b) + f(b,c) = f(ac, b^2) + 1$ for any positive integers a, b, c. Assume there exists a positive integer n such that $f(n,m) \leq f(n, m+1)$ for all positive integers m. Determine all possible values of $f(2025, 2025)$.

11.3 We have n chips that are initially placed on the number line at position 0. On each move, we select a position $x \in \mathbb{Z}$ where there are at least two chips; we take two of these chips, then place one at $x-1$ and the other at $x+1$.

a) Prove that after a finite number of moves, regardless of how the moves are chosen, we will reach a final position where no two chips occupy the same number on the number line.

b) For every possible final position, let Δ represent the difference between the numbers where the rightmost and the leftmost chips are located. Find all possible values of Δ in terms of n.

11.4 Let A be a set of 2025 non-negative integers and $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be a function with the following two properties:

1) For every two distinct positive integers x, y there exists $a \in A$, such that $x - y$ divides $f(x+a) - f(y+a)$.

2) For every positive integer N there exists a positive integer t such that $f(x) \neq f(y)$ whenever $x, y \in [t, t+N]$ are distinct.

Prove that there are infinitely many primes p such that p divides $f(x)$ for some positive integer

x.

12.1 Let a, b, c be positive real numbers with $a + b > c$. Prove that $ax + \sin(bx) + \cos(cx) > 1$ for all $x \in \left(0, \frac{\pi}{a+b+c}\right)$.

12.2 In the plane are fixed two internally tangent circles ω and Ω, so that ω is inside Ω. Denote their common point by T. The point $A \neq T$ moves on Ω and point B on Ω is such that AB is tangent to ω. The line through B, perpendicular to AB, meets the external angle bisector of $\angle ATB$ at P. Prove that, as A varies on Ω, the line AP passes through a fixed point.

12.3 Determine all functions $f : \mathbb{Z}_{\geq 2025} \to \mathbb{Z}_{>0}$ such that $mn+1$ divides $f(m)f(n)+1$ for any integers $m, n \geq 2025$ and there exists a polynomial P with integer coefficients, such that $f(n) \leq P(n)$ for all $n \geq 2025$.

12.4 Prove that a graph containing a copy of each possible tree on n vertices as a subgraph has at least $n(\ln n - 2)$ edges.