Czech-Polish-Slovak Match 2001

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Day 1 June 14th

1 Prove that for any positive numbers $a_1, \dots, a_n$ $(n \geq 2)$

$$(a_1^3+1)(a_2^3+1)\cdots(a_n^3+1) \geq (a_1^2a_2+1)(a_2^2a_3+1)\cdots(a_n^2a_1+1)$$

2 A triangle ABC has acute angles at A and B. Isosceles triangles ACD and BCE with bases AC and BC are constructed externally to triangle ABC such that $\angle ADC = \angle ABC$ and $\angle BEC = \angle BAC$. Let S be the circumcenter of $\triangle ABC$. Prove that the length of the polygonal line DSE equals the perimeter of triangle ABC if and only if $\angle ACB$ is right.

3 Let n and k be positive integers such that $\frac{1}{2}n < k \leq \frac{2}{3}n$. Find the least number m for which it is possible to place m pawns on m squares of an $n \times n$ chessboard so that no column or row contains a block of k adjacent unoccupied squares.

Day 2 June 15th

4 Distinct points A and B are given on the plane. Consider all triangles ABC in this plane on whose sides BC, CA points D, E respectively can be taken so that
(i) $\frac{BD}{BC} = \frac{CE}{CA} = \frac{1}{3}$;
(ii) points A, B, D, E lie on a circle in this order.

Find the locus of the intersection points of lines AD and BE.

5 Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x^2+y) + f(f(x)-y) = 2f(f(x)) + 2y^2 \quad \text{for all } x, y \in \mathbb{R}.$$

6 Points with integer coordinates in cartesian space are called lattice points. We color 2000 lattice points blue and 2000 other lattice points red in such a way that no two blue-red segments have a common interior point (a segment is blue-red if its two endpoints are colored blue and red). Consider the smallest rectangular parallelepiped that covers all the colored points.
(a) Prove that this rectangular parallelepiped covers at least $500,000$ lattice points.
(b) Give an example of a coloring for which the considered rectangular paralellepiped covers at most $8,000,000$ lattice points.