Czech-Polish-Slovak Match 2003

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Day 1 June 16th

1 Given an integer $n \geq 2$, solve in real numbers the system of equations

$$\begin{aligned} \max\{1, x_1\} &= x_2 \\ \max\{2, x_2\} &= 2x_3 \\ &\cdots \\ \max\{n, x_n\} &= nx_1. \end{aligned}$$

2 In an acute-angled triangle ABC the angle at B is greater than 45°. Points D, E, F are the feet of the altitudes from A, B, C respectively, and K is the point on segment AF such that $\angle DKF = \angle KEF$.
(a) Show that such a point K always exists.
(b) Prove that $KD^2 = FD^2 + AF \cdot BF$.

3 Numbers p, q, r lies in the interval $(\frac{2}{5}, \frac{5}{2})$ nad satisfy $pqr = 1$. Prove that there exist two triangles of the same area, one with the sides a, b, c and the other with the sides pa, qb, rc.

Day 2 June 17th

4 Point P lies on the median from vertex C of a triangle ABC. Line AP meets BC at X, and line BP meets AC at Y. Prove that if quadrilateral $ABXY$ is cyclic, then triangle ABC is isosceles.

5 Consider the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $(k = 1, 2, \ldots n-1)$. Determine all positive integers n for which $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$ are all even numbers.

6 Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the condition

$$f(f(x) + y) = 2x + f(f(y) - x) \quad \text{for all } x, y \in \mathbb{R}.$$