Czech-Polish-Slovak Match 2004

www.artofproblemsolving.com/community/c4190

by gauss202, djb86, Beat

Day 1 June 21st

1 Show that real numbers, p, q, r satisfy the condition $p^4(q-r)^2 + 2p^2(q+r) + 1 = p^4$ if and only if the quadratic equations $x^2 + px + q = 0$ and $y^2 - py + r = 0$ have real roots (not necessarily distinct) which can be labeled by x_1, x_2 and y_1, y_2, respectively, in such a way that $x_1y_1 - x_2y_2 = 1$.

2 Show that for each natural number k there exist only finitely many triples (p, q, r) of distinct primes for which p divides $qr - k$, q divides $pr - k$, and r divides $pq - k$.

3 A point P in the interior of a cyclic quadrilateral $ABCD$ satisfies $\angle BPC = \angle BAP + \angle PDC$. Denote by E, F and G the feet of the perpendiculars from P to the lines AB, AD and DC, respectively. Show that the triangles FEG and PBC are similar.

Day 2 June 22nd

4 Solve in real numbers the system of equations:

$$\frac{1}{xy} = \frac{x}{z} + 1$$
$$\frac{1}{yz} = \frac{y}{x} + 1$$
$$\frac{1}{zx} = \frac{z}{y} + 1$$

5 Points K, L, M on the sides AB, BC, CA respectively of a triangle ABC satisfy $\frac{AK}{KB} = \frac{BL}{LC} = \frac{CM}{MA}$. Show that the triangles ABC and KLM have a common orthocenter if and only if $\triangle ABC$ is equilateral.

6 On the table there are $k \geq 3$ heaps of $1, 2, \dots, k$ stones. In the first step, we choose any three of the heaps, merge them into a single new heap, and remove 1 stone from this new heap. Thereafter, in the i-th step ($i \geq 2$) we merge some three heaps containing more than i stones in total and remove i stones from the new heap. Assume that after a number of steps a single heap of p stones remains on the table. Show that the number p is a perfect square if and only if so are both $2k + 2$ and $3k + 1$. Find the least k with this property.