Czech-Polish-Slovak Match 2007

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1 Find all polynomials P with real coefficients satisfying $P(x^2) = P(x) \cdot P(x+2)$ for all real numbers x.

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2 The Fibonacci sequence is defined by $a_1 = a_2 = 1$ and $a_{k+2} = a_{k+1} + a_k$ for $k \in \mathbb{N}$. Prove that for any natural number m, there exists an index k such that $a_k^4 - a_k - 2$ is divisible by m.

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3 A convex quadrilateral $ABCD$ inscribed in a circle k has the property that the rays DA and CB meet at a point E for which $CD^2 = AD \cdot ED$. The perpendicular to ED at A intersects k again at point F. Prove that the segments AD and CF are congruent if and only if the circumcenterof $\triangle ABE$ lies on ED.

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4 For any real number $p \geq 1$ consider the set of all real numbers x with

$$p < x < \left(2 + \sqrt{p + \frac{1}{4}}\right)^2.$$

Prove that from any such set one can select four mutually distinct natural numbers a, b, c, d with $ab = cd$.

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5 For which $n \in \{3900, 3901, \cdots, 3909\}$ can the set $\{1, 2, ..., n\}$ be partitioned into (disjoint) triples in such a way that in each triple one of the numbers equals the sum of the other two?

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6 Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that $\angle PAB + \angle PDC \leq 90^\circ$ and $\angle PBA + \angle PCD \leq 90^\circ$. Prove that $AB + CD \geq BC + AD$.
