Czech-Polish-Slovak Match 2008

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by Sayan

Day 1 June 22nd

1 Determine all triples (x, y, z) of positive real numbers which satisfies the following system of equations

$$2x^3 = 2y(x^2+1) - (z^2+1),$$
$$2y^4 = 3z(y^2+1) - 2(x^2+1),$$
$$2z^5 = 4x(z^2+1) - 3(y^2+1).$$

2 $ABCDEF$ is a convex hexagon, such that $|\angle FAB| = |\angle BCD| = |\angle DEF|$ and $|AB| = |BC|$, $|CD| = |DE|$, $|EF| = |FA|$. Prove that the lines AD, BE and CF are concurrent.

3 Find all primes p such that the expression

$$\binom{p}{1}^2 + \binom{p}{2}^2 + \cdots + \binom{p}{p-1}^2$$

is divisible by p^3.

Day 2 June 25th

1 Prove that there exists a positive integer n, such that the number $k^2 + k + n$ does not have a prime divisor less than 2008 for any integer k.

2 $ABCDE$ is a regular pentagon. Determine the smallest value of the expression

$$\frac{|PA| + |PB|}{|PC| + |PD| + |PE|},$$

where P is an arbitrary point lying in the plane of the pentagon $ABCDE$.

3 Find all triplets (k, m, n) of positive integers having the following property: Square with side length m can be divided into several rectangles of size $1 \times k$ and a square with side length n.