Czech-Polish-Slovak Match 2009

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Day 1

1 Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ that satisfy

$$\Big(1+yf(x)\Big)\Big(1-yf(x+y)\Big)=1$$

for all $x,y\in\mathbb{R}^+$.

2 For positive integers a and k, define the sequence $a_1,a_2,\ldots$ by

$$a_1=a,\qquad \text{and} \qquad a_{n+1}=a_n+k\cdot\varrho(a_n)\qquad \text{for } n=1,2,\ldots$$

where $\varrho(m)$ denotes the product of the decimal digits of m (for example, $\varrho(413)=12$ and $\varrho(308)=0$). Prove that there are positive integers a and k for which the sequence $a_1,a_2,\ldots$ contains exactly 2009 different numbers.

3 Let ω denote the excircle tangent to side BC of triangle ABC. A line ℓ parallel to BC meets sides AB and AC at points D and E, respectively. Let ω' denote the incircle of triangle ADE. The tangent from D to ω (different from line AB) and the tangent from E to ω (different from line AC) meet at point P. The tangent from B to ω' (different from line AB) and the tangent from C to ω' (different from line AC) meet at point Q. Prove that, independent of the choice of ℓ, there is a fixed point that line PQ always passes through.

Day 2

4 Given a circle, let AB be a chord that is not a diameter, and let C be a point on the longer arc AB. Let K and L denote the reflections of A and B, respectively, about lines BC and AC, respectively. Prove that the distance between the midpoint of AB and the midpoint of KL is independent of the choice of C.

5 The n-tuple $(a_1,a_2,\ldots,a_n)$ of integers satisfies the following:
(i) $1\le a_1<a_2<\cdots<a_n\le 50$
(ii) for each n-tuple $(b_1,b_2,\ldots,b_n)$ of positive integers, there exist a positive integer m and an n-tuple $(c_1,c_2,\ldots,c_n)$ of positive integers such that

$$mb_i=c_i^{a_i}\qquad \text{for } i=1,2,\ldots,n.$$

Prove that $n\le 16$ and determine the number of n-tuples $(a_1,a_2,\ldots,a_n)$ satisfying these conditions for $n=16$.

6 Let $n \geq 16$ be an integer, and consider the set of n^2 points in the plane:

$$G = \big\{(x, y) \mid x, y \in \{1, 2, \dots, n\}\big\}.$$

Let A be a subset of G with at least $4n\sqrt{n}$ elements. Prove that there are at least n^2 convex quadrilaterals whose vertices are in A and all of whose diagonals pass through a fixed point.