Czech-Polish-Slovak Match 2010

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by Shu

Day 1

1 Find all triples (a,b,c) of positive real numbers satisfying the system of equations

$$a\sqrt{b}-c\& =a, \qquad b\sqrt{c}-a\& =b, \qquad c\sqrt{a}-b\& =c.$$

2 Given any 60 points on a circle of radius 1, prove that there is a point on the circle the sum of whose distances to these 60 points is at most 80.

3 Let p be a prime number. Prove that from a $p^2\times p^2$ array of squares, we can select p^3 of the squares such that the centers of any four of the selected squares are not the vertices of a rectangle with sides parallel to the edges of the array.

Day 2

1 Given any collection of 2010 nondegenerate triangles, their sides are painted so that each triangle has one red side, one blue side, and one white side. For each color, arrange the side lengths in order: -let $b_1\le b_2\le\cdots\le b_{2011}$ denote the lengths of the blue sides;
-let $r_1\le r_2\le\cdots\le r_{2011}$ denote the lengths of the red sides; and
-let $w_1\le w_2\le\cdots\le w_{2011}$ denote the lengths of the white sides. Find the largest integer k for which there necessarily exists at least k indices j such that b_j, r_j, w_j are the side lengths of a nondegenerate triangle.

2 Let x, y, z be positive real numbers satisfying $x+y+z\ge 6$. Find, with proof, the minimum value of

$$x^2+y^2+z^2+\frac{x}{y^2+z+1}+\frac{y}{z^2+x+1}+\frac{z}{x^2+y+1}.$$

3 Let $ABCD$ be a convex quadrilateral for which

$$AB+CD=\sqrt{2}\cdot AC \qquad \text{and} \qquad BC+DA=\sqrt{2}\cdot BD.$$

Prove that $ABCD$ is a parallelogram.