Czech-Polish-Slovak Match 2011

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by Shu

Day 1 June 19th

1 Let a, b, c be positive real numbers satisfying $a^2 < bc$. Prove that $b^3 + ac^2 > ab(a + c)$.

2 Written on a blackboard are n nonnegative integers whose greatest common divisor is 1. A *move* consists of erasing two numbers x and y, where $x \geq y$, on the blackboard and replacing them with the numbers $x - y$ and $2y$. Determine for which original n-tuples of numbers on the blackboard is it possible to reach a point, after some number of moves, where $n - 1$ of the numbers of the blackboard are zeroes.

3 Points A, B, C, D lie on a circle (in that order) where AB and CD are not parallel. The length of arc AB (which contains the points D and C) is twice the length of arc CD (which does not contain the points A and B). Let E be a point where $AC = AE$ and $BD = BE$. Prove that if the perpendicular line from point E to the line AB passes through the center of the arc CD (which does not contain the points A and B), then $\angle ACB = 108^\circ$.

Day 2 June 22nd

1 A polynomial $P(x)$ with integer coefficients satisfies the following: if $F(x)$, $G(x)$, and $Q(x)$ are polynomials with integer coefficients satisfying $P\Big(Q(x)\Big) = F(x) \cdot G(x)$, then $F(x)$ or $G(x)$ is a constant polynomial. Prove that $P(x)$ is a constant polynomial.

2 In convex quadrilateral $ABCD$, let M and N denote the midpoints of sides AD and BC, respectively. On sides AB and CD are points K and L, respectively, such that $\angle MKA = \angle NLC$. Prove that if lines BD, KM, and LN are concurrent, then

$$\angle KMN = \angle BDC \qquad \text{and} \qquad \angle LNM = \angle ABD.$$

3 Let a be any integer. Prove that there are infinitely many primes p such that

$$p \,|\, n^2 + 3 \qquad \text{and} \qquad p \,|\, m^3 - a$$

for some integers n and m.