**Czech-Polish-Slovak Match 2012**

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by Sayan

**Day 1**   June 24th

**1**   Given a positive integer $n$, let $\tau(n)$ denote the number of positive divisors of $n$ and $\varphi(n)$ denote the number of positive integers not exceeding $n$ that are relatively prime to $n$. Find all $n$ for which one of the three numbers $n, \tau(n), \varphi(n)$ is the arithmetic mean of the other two.

**2**   Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + f(y)) - f(x) = (x + f(y))^4 - x^4$$

for all $x, y \in \mathbb{R}$.

**3**   Let $ABCD$ be a cyclic quadrilateral with circumcircle $\omega$. Let $I, J$ and $K$ be the incentres of the triangles $ABC, ACD$ and $ABD$ respectively. Let $E$ be the midpoint of the arc $DB$ of circle $\omega$ containing the point $A$. The line $EK$ intersects again the circle $\omega$ at point $F$ $(F \neq E)$. Prove that the points $C, F, I, J$ lie on a circle.

**Day 2**   June 27th

**1**   Let $ABC$ be a right angled triangle with hypotenuse $AB$ and $P$ be a point on the shorter arc $AC$ of the circumcircle of triangle $ABC$. The line, perpendicuar to $CP$ and passing through $C$, intersects $AP, BP$ at points $K$ and $L$ respectively. Prove that the ratio of area of triangles $BKL$ and $ACP$ is independent of the position of point $P$.

**2**   City of Mar del Plata is a square shaped $WSEN$ land with $2(n + 1)$ streets that divides it into $n \times n$ blocks, where $n$ is an even number (the leading streets form the perimeter of the square). Each block has a dimension of $100 \times 100$ meters. All streets in Mar del Plata are one-way. The streets which are parallel and adjacent to each other are directed in opposite direction. Street $WS$ is driven in the direction from $W$ to $S$ and the street $WN$ travels from $W$ to $N$. A street cleaning car starts from point $W$. The driver wants to go to the point $E$ and in doing so, he must cross as much as possible roads. What is the length of the longest route he can go, if any $100$-meter stretch cannot be crossed more than once? (The figure shows a plan of the city for $n = 6$ and one of the possible - but not the longest - routes of the street cleaning car. See http://goo.gl/maps/JAzD too.)
`http://s14.postimg.org/avfg7ygb5/CPS_2012_P5.jpg`

**3**   Let $a, b, c, d$ be positive real numbers such that $abcd = 4$ and

$$a^2 + b^2 + c^2 + d^2 = 10.$$

Find the maximum possible value of $ab + bc + cd + da$.