IMC 2008

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–	Day 1
1	Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $$f(x) - f(y) \in \mathbb{Q} \quad \text{for all} \quad x - y \in \mathbb{Q}$$
2	Denote by $\mathbb{V}$ the real vector space of all real polynomials in one variable, and let $\gamma : \mathbb{V} \to \mathbb{R}$ be a linear map. Suppose that for all $f, g \in \mathbb{V}$ with $\gamma(fg) = 0$ we have $\gamma(f) = 0$ or $\gamma(g) = 0$. Prove that there exist $c, x_0 \in \mathbb{R}$ such that $$\gamma(f) = cf(x_0) \quad \forall f \in \mathbb{V}$$
3	Let p be a polynomial with integer coecients and let $a_1 < a_2 < \cdots < a_k$ be integers. Given that $p(a_i) \neq 0 \forall\ i = 1, 2, \cdots, k$. (a) Prove $\exists\ a \in \mathbb{Z}$ such that $$p(a_i) \mid p(a) \ \ \forall i = 1, 2, \dots, k$$ (b) Does there exist $a \in \mathbb{Z}$ such that $$\prod_{i=1}^{k} p(a_i) \mid p(a)$$
4	We say a triple of real numbers (a_1, a_2, a_3) is **better** than another triple (b_1, b_2, b_3) when exactly two out of the three following inequalities hold: $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$. We call a triple of real numbers **special** when they are nonnegative and their sum is 1. For which natural numbers n does there exist a collection S of special triples, with $\lvert S \rvert = n$, such that any special triple is bettered by at least one element of S?
5	Does there exist a finite group G with a normal subgroup H such that $\lvert \text{Aut } H \rvert > \lvert \text{Aut } G \rvert$? Disprove or provide an example. Here the notation $\lvert \text{Aut } X \rvert$ for some group X denotes the number of isomorphisms from X to itself.

6 For a permutation $\sigma \in S_n$ with $(1,2,\dots,n) \mapsto (i_1,i_2,\dots,i_n)$, define

$$D(\sigma) = \sum_{k=1}^{n} |i_k - k|$$

Let

$$Q(n,d) = |\{\sigma \in S_n : D(\sigma) = d\}|$$

Show that when $d \geq 2n$, $Q(n,d)$ is an even number.

– Day 2

1 Let n, k be positive integers and suppose that the polynomial $x^{2k} - x^k + 1$ divides $x^{2n} + x^n + 1$. Prove that $x^{2k} + x^k + 1$ divides $x^{2n} + x^n + 1$.

2 Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

3 Let n be a positive integer. Prove that 2^{n-1} divides

$$\sum_{0 \leq k < n/2} \binom{n}{2k+1} 5^k.$$

4 Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients, and let $f(x), g(x) \in \mathbb{Z}[x]$ be nonconstant polynomials such that $g(x)$ divides $f(x)$ in $\mathbb{Z}[x]$. Prove that if the polynomial $f(x) - 2008$ has at least 81 distinct integer roots, then the degree of $g(x)$ is greater than 5.

5 Let n be a positive integer, and consider the matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ where $a_{ij} = 1$ if $i + j$ is prime and $a_{ij} = 0$ otherwise.
Prove that $|\det A| = k^2$ for some integer k.

6 Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, let $d > 0$, and suppose that S is a set of points (not necessarily countable) in $\mathcal{H}$ such that the distance between any two distinct points in S is equal to d. Show that there is a point $y \in \mathcal{H}$ such that

$$\left\{ \frac{\sqrt{2}}{d}(x - y) : \ x \in S \right\}$$

is an orthonormal system of vectors in $\mathcal{H}$.